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SHORT COMMUNICATIONS

Quadratic Sufficient Conditions for Strong Minimality of Abnormal Sub-Riemannian Geodesics

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Abstract. The minimizing problem for the length of trajectories with respect to a submetric on a distribution is considered. Quadratic sufficient conditions for the strong minimality of abnormal trajectories of arbitrary length are obtained. The results hold for distributions of arbitrary dimensions and for a broad class of submetrics, including those of sub-Riemannian and sub-Finsler metrics.

INTRODUCTION

Let $\mathcal{M}$ be an open set in $\mathbb{R}^n$, let k vector fields $r_0(x), \dots, r_{k-1}(x)$, be given on $\mathcal{M}$, and let these fields be C^2 smooth on $\mathcal{M}$ and linear independent at any point $x \in \mathcal{M}$. These fields define a k-dimensional distribution

$$\Gamma(x) = \text{ Lin } \{r_0(x), \dots, r_{k-1}(x)\}$$

on $\mathcal{M}$. A real function $q(x, \bar{x})$ of $x \in \mathcal{M}$ and $\bar{x} \in \Gamma(x)$ is called *a submetric* on $\Gamma(x)$ if, for any chosen $x \in \mathcal{M}$, this function defines a positive sublinear functional of $\bar{x}$. We assume that q is continuous with respect to $(x, \bar{x})$. In particular, if $q^2(x, \bar{x}) = (Q(x)\bar{x}, \bar{x})$ is a positive-definite quadratic form of $\bar{x} \in \Gamma(x)$, then q is called a *sub-Riemannian metric*, and $\Gamma(x)$ together with q define a sub-Riemannian structure on $\mathcal{M}$. A more general case (which is still special) is related to a sub-Finsler metric for which $q(x, -\bar{x}) = q(x, \bar{x})$.

Representing vectors $\bar{x} \in \Gamma(x)$ in the form

$$\bar{x} = \sum u_i\, r_i(x),\ \ u = (u_0, \dots, u_{k-1}) \in \mathbb{R}^k,$$

we can define the function

$$\varphi(x, u) = q(x, \bar{x}) = q(x, \sum u_i\, r_i(x)).$$

Let us consider the absolutely continuous curves $x(t)$ in $\mathcal{M}$, $t \in [0, T]$, such that $\dot{x}(t) \in \Gamma(x(t))$ a.e. on [0, T]. These curves are said to be Γ-*admissible*, or simply *admissible* because Γ is the same throughout the paper. These curves can also be defined as solutions of the differential equation

$$\dot{x}(t) = \sum u_i(t)\, r_i(x(t)), \tag{1}$$

where $u_i \in L_1[0, T]$, $i = 0, \dots, k-1$. For any Γ-admissible curve, one can define its length by the formula

$$J(x) = \int_0^T q(x, \dot{x})\, dt$$

or, using the presentation (1), by the relation

$$J(x) = \int_0^T \varphi(x, u) dt.$$

Now let $\hat{x}(t)$, $t \in [0,T]$, be a Γ-admissible curve joining two given points $\hat{x}(0) = p_0$ and $\hat{x}(T) = p_1$. The question is: What are necessary and sufficient conditions for this curve to have minimal length among all admissible curves joining p_0 and p_1 and (in a sense) close to $\hat{x}(t)$? This question is the most difficult in the case of the so-called *abnormal* curves. As well as the related question of the so-called *rigidity*, this problem has recently attracted the serious attention of experts, and it is precisely the sufficient conditions that were mainly studied (see [1–7] and the references therein). However, even the problem of sufficient minimality conditions is not solved completely. Namely,

a) only the case of $\dim \Gamma = 2$ was treated,

b) only sub-Riemannian metrics were considered,

c) there are no conditions for a strong minimum, in the sense of the classical calculus of variations (CCV) (the "strong" minimum discussed in [6] is weaker than the classical strong minimum, and in fact it is the so-called Pontryagin minimum; see below and, e.g., [7, 10–14]),

d) the sufficient conditions in [1, 4, 5] guarantee the minimality only for small pieces of a curve,

e) the sufficient conditions in [6] are very restrictive and can be weakened.

Although the minimum problem for lengths is a special case of an extremal problem, we stress that the known minimality conditions, except for Pontryagin's maximum principle (PMP), are not applications of general theory to this special problem but follow from *ad hoc* theories and considerations. We approach this problem by using the theory of quadratic conditions for local minimum for singular extremals in optimal control problems (this theory was developed by Milyutin [8] and the author [9–14]), and our results are stronger and more general than the previously known ones. In the present paper we discuss only *sufficient* minimality conditions.

BASIC NOTIONS AND ASSUMPTIONS

The problem of determining a curve minimizing length can be stated as an optimal control problem on a fixed time interval [0, T] as follows.

Problem (Z):

$$\dot{x} = z \sum u_i\, r_i(x), \qquad \dot{z} = 0, \qquad z > 0, \qquad x(0) = p_0, \quad x(T) = p_1,$$
$$\varphi(x,u) \leqslant 1, \tag{2}$$
$$J = z(0) \longrightarrow \min .$$

Here z is a positive constant bounding the velocity, $\|\dot{x}\| \leqslant z \cdot \varphi(x,u) \leqslant z$, and the problem is to come from p_0 to p_1 at a given time T with the minimal possible upper bound of the speed. Note that, in general, Problem (Z) includes the mixed state-control constraint (2), i.e., the admissible control set depends on x: $U(x) = \{u \mid \varphi(x,u) \leqslant 1\}$. For sub-Riemannian metrics this constraint can be reduced (in an appropriate basis) to the form $|u| \leqslant 1$, and thus Problem (Z) becomes a classical optimal control problem of Pontryagin type.

Let $\hat{w} = (\hat{z}, \hat{x}, \hat{u})$ be an admissible trajectory of Problem (Z). According to CCV, the trajectory $\hat{w}$ is called a *point of the weak minimum* (or a *weak minimizer*) of Problem (Z) if it is a point of local minimum with respect to the norm $\|w\| = |z| + \|x\|_C + \|u\|_\infty$, and $\hat{w}$ is called a *point of strong minimum* (or a *strong minimizer*) of Problem (Z) if it is a point of local minimum with respect to the seminorm given by $\|w\|' = |z| + \|x\|_C$ for arbitrary u.

Throughout the paper we assume that a trajectory $\hat{x}(t)$ under consideration satisfies the following condition.

Assumption A1. The curve $\hat{x}(t)$, $t \in [0,T]$, is C^3 smooth with nonzero derivative and without self-intersections.

In fact, the last requirement is unessential because one can readily get rid of it, but we admit it here for simplicity. (In [1–6] it is actually assumed that the curve under consideration is C^∞-smooth. However, we do not concentrate on weakening of the smoothness assumptions.)

Denote by $\hat{\chi}$ the image of the curve $\hat{x}(t)$, $t \in [0,T]$, in the space $\mathbb{R}^n$.

Proposition. *A trajectory $\hat{w}$ is a strong minimizer of Problem* (Z) *if and only if the corresponding curve $\hat{x}(t)$ has minimal length among all admissible curves $x(t)$ joining p_0 and p_1 and lying in a neighborhood of the set $\hat{\chi}$.*

This means that the notion of strong minimum in Problem (Z) agrees with geometric sense.

Definition 1. The trajectory $\hat{w}$ is said to be *abnormal* if there exists a Lipschitzian n-vector function (more precisely, an n-covector function) $\psi(t)$ such that

$$-\dot{\psi} \;=\; \hat{z}\,\psi \sum_{i=0}^{k-1} \hat{u}_i\, r_i'(\hat{x}), \qquad t \in [0,T], \tag{3}$$

$$\psi(t)\, r_i(\hat{x}(t)) = 0, \qquad t \in [0,T], \quad i = 0,1,\dots,k-1, \quad \text{i.e.,} \quad \psi(t) \perp \Gamma(\hat{x}(t)); \tag{4}$$

ψ is said to be an *adjoint* or *costate variable.* (The prime at a function stands for the derivative of this function with respect to x unless otherwise stated. Given an n-covector ψ and an n-vector r, the expression (ψ, r) or simply ψr denotes the value of the linear functional ψ at r.)

The set of all functions $\psi(t)$, satisfying (3) and (4) and normalized by the condition $|\psi(0)| = 1$ is denoted by $\Psi_0 = \Psi_0(\hat{w})$. Thus, $\hat{w}$ is abnormal if and only if $\Psi_0(\hat{w})$ is nonempty. The abnormal trajectories obviously satisfy PMP (which we do not write out here). The set $\Psi_0(\hat{w})$ does not depend on the choice of the submetric and, according to [7], on the choice of a basis in $\Gamma(x)$.

Definition 2. An abnormal trajectory $\hat{w}$ is said to be *strictly abnormal*, or *singular*, if it satisfies PMP for $\psi \in \Psi_0(\hat{w})$ only. The trajectories satisfying PMP and such that $\Psi_0(\hat{w}) = \varnothing$ are said to be *normal* stationary trajectories.

For normal trajectories, the problem of their strong minimality is rather simple because, in this case, the second variation of the Lagrange function contains $\varphi''_{uu}(\hat{x},\hat{u})$, and, under the conventional assumption that this matrix is positive definite on the subspace $\varphi'_u(\hat{x},\hat{u})\,\bar{u} = 0$, one can use the known conditions of CCV (cf. [1] for sub-Riemannian metrics). The problem is much more difficult for abnormal trajectories, which we study in this paper.

We admit some assumptions on the submetric.

Definition 3. We say that a submetric $q(x,\bar{x})$ has a C^2 *smooth* (*hyper-*) *plane of support in a neighborhood of the curve* $\hat{x}(t)$ if in a neighborhood of $\hat{\chi}$ there exist a C^2 smooth $(k-1)$-dimensional subspace $\Gamma_0(x) \subset \Gamma(x)$ and a C^2 smooth nonzero vector field $r_0(x) \in \Gamma(x)$ such that the affine hyperplane $r_0(x) + \Gamma_0(x)$ and the relative interior of the hodograph

$$F(x) = \{\bar{x} \in \Gamma(x) \mid q(x,\bar{x}) \leqslant q(x, r_0(x))\},$$

are disjoint, and $r_0(\hat{x}(t)) = \dot{\hat{x}}(t)$ for all t.

By using a basis in $\Gamma(x)$, this property can be reformulated as follows: in a neighborhood of $\hat{\chi}$, there exist C^2smooth nonzero vector functions $l(x)$, $v(x) \in \mathbb{R}^k$ such that, if $u \in \mathbb{R}^k$ and $\varphi(x,u) \leqslant \varphi(x, v(x))$, then $(l(x), u) \leqslant (l(x), v(x))$, and $v(\hat{x}(t)) = \hat{u}(t)$ for all t.

We can readily see that any C^3 smooth metric q on $\mathcal{M} \times \mathbb{R}^n$ (for $\bar{x} \neq 0$) and any C^2 smooth Riemannian metric restricted on any C^2 smooth distribution $\Gamma(x)$ have a C^2 smooth plane of support in a neighborhood of any admissible curve $\hat{x}(t)$ satisfying Assumption A1. Below we omit the words "C^2 smooth" for brevity.

Definition 4. A hyperplane of support $\Gamma_0(x)$ in a neighborhood of $\hat{x}(t)$ is said to be *strict* if, for any x in a neighborhood of $\hat{\chi}$, the affine hyperplane $r_0(x) + \Gamma_0(x)$ has the unique common point $r_0(x)$ with the hodograph $F(x)$.

Obviously, if a submetric is strictly convex (i.e., for any x, the sublinear functional $q(x,\bar{x})$ is strictly convex with respect to $\bar{x}$), then any hyperplane of support is automatically strict. In particular, this is the case for any sub-Riemannian metric.

Assumption A2. The submetric q has a C^2 smooth strict hyperplane of support in a neighborhood of $\hat{x}(t)$.

Note that the validity of this assumption does not depend on the choice of a parametrization of the curve $\hat{x}$. Any sub-Riemannian metric satisfies Assumption A2.

We take $\varphi(\hat{x},\hat{u}) = 1$ on [0,T] (otherwise one can reparametrize $\hat{x}$ and reduce z, and hence $\hat{w}$ cannot be optimal) and $\hat{z} = 1$, so that T is the length of the curve $\hat{x}$. Now let us choose a special basis in $\Gamma(x)$.

Definition 5. Following [7], a basis in $\Gamma(x)$ is said to be *adjoint* for the trajectory $\hat{x}(t)$ if $r_0(\hat{x}(t)) = \dot{\hat{x}}(t)$ on $[0,T]$.

In PZ written in the adjoint basis, the reference control is $\hat{u} = (1,0,\dots,0)$. The state components $\hat{z} \equiv 1$ and $\hat{x}(t)$ do not depend on the choice of a basis in $\Gamma(x)$.

The adjoint equation (3) in the adjoint basis becomes

$$\dot{\psi}(t) \;=\; -\psi(t)\, r_0'(\hat{x}(t))\,. \tag{5}$$

Along with Problem (Z), we consider the following System (R) in the adjoint basis:

$$\dot{x} \;=\; z\, r_0(x) \;+\; \sum_{i=1}^{k-1} u_i\, r_i(x), \qquad \dot{z} \;=\; 0, \qquad t\in[0,T], \qquad x(0)=p_0, \qquad x(T)=p_1. \tag{6}$$

This differs from Problem (Z) on the following points: the cost functional J is absent, only the term with $i=0$ is multiplied by z, and it is assumed that $u_0 = 1$ and the other controls u_i are free. The number of controls becomes equal to $k-1$. The submetric q does not enter System (R).

This system appears in the study of the rigidity of the curve $\hat{x}(t)$ in the class of all admissible curves joining the points p_0 and p_1 [7]. Namely, $\hat{x}$ is rigid if and only if the triple $\hat{w} = (\hat{z},\hat{x},\hat{u})$ is isolated in the set of all trajectories of System (R) with respect to the norm

$$\|w\| \;=\; |z| + \|x\|_C + \|u\|_\infty.$$

The abnormality of $\hat{w}$ in Problem (Z) can be treated as the absence of controllability in System (R) at $\hat{w}$, so that $\Psi_0(\hat{w})$ is nonempty if and only if the problem is not controllable. For System (R), this set was investigated by Milyutin in [7].

SECOND VARIATIONS ON THE CRITICAL SUBSPACE

For any function $\psi(t)$, let us consider the Lagrange function for System (R),

$$\Phi[\psi](z,x,u) \;=\; \psi_0\, x_0 \;-\; \psi_T\, x_T \;+\; \int_0^T \psi\,\Big(\dot{x} \;-\; z\, r_0(x) \;-\; \sum_{i=1}^{k-1} u_i\, r_i(x)\Big)\, dt$$

and a half of its second variation at $\hat{w}$, $\Omega\,[\psi](\bar{w})$, which must be considered on the subspace $\mathcal{K}$ of critical variations $\bar{w} = (\bar{z},\bar{x},\bar{u})$,

$$\dot{\bar{x}} \;=\; \bar{z}\, r_0(\hat{x}) \;+\; r_0'(\hat{x})\,\bar{x} \;+\; \sum_{i=1}^{k-1} \bar{u}_i\, r_i(\hat{x}), \qquad \dot{\bar{z}} \;=\; 0, \qquad \bar{x}(0)=0, \qquad \bar{x}(T)=0. \tag{7}$$

It is convenient to pass from the variables $\bar{w} = (\bar{z},\bar{x},\bar{u})$ to $(\bar{z},\bar{\xi},\bar{y},\bar{u})$, which we denote by the same letter $\bar{w}$, where

$$\bar{y} = (\bar{y}_1,\dots,\bar{y}_{k-1}), \quad \bar{u} = (\bar{u}_1,\dots,\bar{u}_{k-1}), \quad \dot{\bar{y}} = \bar{u}, \quad \bar{y}(0)=0, \quad \bar{\xi} = \bar{x} - \sum_{j=1}^{k-1} \bar{y}_j\, r_j(\hat{x})$$

(the known Goh transformation), and hence the critical subspace consists of all $\bar w = (\bar z, \bar\xi, \bar y, \bar u)$ satisfying the relations

$$\dot{\bar\xi} \;=\; \bar z\, r_0(\hat x) \;+\; r_0'(\hat x)\,\bar\xi \;+\; \sum_{j=1}^{k-1} \bar y_j\,[r_0, r_j], \tag{8}$$

$$\bar\xi(0) = 0, \qquad \bar\xi(T) + \sum \bar y_j(T)\, r_j(\hat x(T)) = 0, \qquad \dot{\bar y} = \bar u, \qquad \bar y(0) = 0. \tag{9}$$

(Here and below $[f, g] = f'g - g'f$ are Lie brackets, the derivatives are evaluated along $\hat x(t)$, and the expressions $r_0'(\hat x)\,\bar\xi$ and $(r_0''(\hat x)\,\bar\xi, \bar\xi)$ stand for the first and second differentials of the mapping r_0 at the point $\hat x(t)$ in the direction $\bar\xi(t)$, respectively.)

Under this transformation, the second variation $\Omega\,[\psi](\bar w)$ becomes [7, 14]

$$\begin{aligned}\Omega\,[\psi](\bar z, \bar\xi, \bar y, \bar u) \;=\; &\frac12 \left\{ \sum_{i,j=1}^{k-1} \bar y_i\, \bar y_j\, \psi\,(r_i'(\hat x)\, r_j(\hat x)) \right\}\Bigg|_T \\ &+ \int_0^T \left(-\frac12\, \psi\,(r_0''\,\bar\xi, \bar\xi) \;-\; \bar z\,\psi\,(r_0'\,\bar\xi) \;+\; \sum_{i=1}^{k-1} \bar y_i\, \psi\,[r_i, r_0]'\,\bar\xi \right. \\ &\left. + \frac12 \sum_{i,j=1}^{k-1} \bar y_i\,\bar y_j\,\psi\,[[r_i, r_0], r_j] \;+\; \frac12 \sum_{i,j=1}^{k-1} \bar y_i\, \bar u_j\, \psi\,[r_i, r_j] \right) dt.\end{aligned} \tag{10}$$

For any subset $M \subset \Psi_0$ we define the functional

$$\Omega\,[M](\bar w) \;=\; \sup_{\psi \in M} \Omega\,[\psi](\bar w).$$

Now let us introduce special subsets of Ψ_0. For any $a \in \mathbb{R}$, denote by $G_a(\Psi_0)$ the set of all $\psi \in \Psi_0$ satisfying the following two conditions [7, 14] along $\hat x(t)$ on [0,T]:

$$\psi(t)\,[r_i, r_j]\,(\hat x(t)) \;=\; 0 \qquad \forall\, i, j = 1, \dots, k-1, \tag{11}$$

$$\sum_{i,j=1}^{k-1} \bar y_i\, \bar y_j\, \psi[[r_i, r_0], r_j] \;\geqslant\; a\,|\,\bar y\,|^2 \qquad \forall\, \bar y = (\bar y_1, \dots, \bar y_{k-1}) \in \mathbb{R}^{k-1}. \tag{12}$$

(Essentially, these are the Goh conditions written in terms of Lie brackets.) Note that if (11) holds, then the last term in (10) vanishes. The submetric q does not enter the definitions of the sets Ψ_0 and $G_a(\Psi_0)$. We also note that Eq. (11) for $j = 0$ obviously follows from (4), because

$$\frac{d}{dt}(\psi\, r_i) = \psi\,[r_i, r_0] = 0.$$

We also write

$$G_+(\Psi_0) \;=\; \bigcup_{a>0} G_a(\Psi_0).$$

Due to [7], this set does not depend on the choice of an adjoint basis for $\hat x(t)$.

Remark 1. If $k = \dim\Gamma(x) = 2$, then the control u in System (R) (and hence in the quadratic form (10)) is a scalar. In this case, condition (11) and condition (16) below hold trivially. The principal features of the multidimensional distribution $\Gamma(x)$ become significant starting from $\dim\Gamma(x) = 3$.

Next, we introduce a quadratic estimate $\gamma(\bar{w})$. According to the general theory [9–13], we must take

$$\gamma(\bar{w}) = |\bar{\xi}(0)|^2 + |\bar{z}|^2 + \sum_{i=1}^{k-1} |\bar{y}_i(T)|^2 + \sum_{i=1}^{k-1} \int_0^T |\bar{y}_i|^2\, dt. \tag{13}$$

However, on the critical subspace $\mathcal{K}$ we have $\bar{\xi}(0) = 0$, and it follows from [15] that

$$|\bar{z}| + |\bar{y}(T)| \leqslant \text{const}\, \|\bar{y}\|_1;$$

hence, the order of γ coincides with that of the last term in (13), i.e., there exist numbers $c_2 \geqslant c_1 > 0$ (one can actually set $c_1 = 1$) such that on $\mathcal{K}$ we have

$$c_1 \int_0^T |\bar{y}|^2\, dt \leqslant \gamma(\bar{w}) \leqslant c_2 \int_0^T |\bar{y}|^2\, dt.$$

Now let us define an important class of quadratically rigid trajectories.

Definition 6. Following [7], a trajectory $\hat{w}$ is said to be *quadratically rigid* if $\Psi_0(\hat{w}) \neq \varnothing$ and if there exists $a > 0$ such that

$$\Omega\,[\Psi_0](\bar{w}) \geqslant a \int_0^T |\bar{y}|^2\, dt \qquad \forall\, \bar{w} \in \mathcal{K}. \tag{14}$$

It can be shown [9] that Eq. (14) is equivalent to the condition that the set $G_a(\Psi_0)$ is nonempty for the same a and

$$\Omega\,[G_a(\Psi_0)](\bar{w}) \geqslant a \int_0^T |\bar{y}|^2\, dt \qquad \forall\, \bar{w} \in \mathcal{K}. \tag{15}$$

In [7] Milyutin proved that condition (14) (or (15)) is sufficient for rigidity, and that this condition does not depend on the choice of the adjoint basis, and thus one can speak of the quadratic rigidity of the curve $\hat{x}(t)$. Milyutin also proved that this condition does not depend on the parametrization of the curve $\hat{x}$.

Now we are ready to state the main results of the paper.

QUADRATIC SUFFICIENT CONDITIONS

Theorem 1 (Sufficient conditions for strong minimality). *Let $\hat{x}(t)$ be a quadratically rigid trajectory. Then, for any submetric on $\Gamma(x)$ with a strict hyperplane of support in a neighborhood of $\hat{x}(t)$, this trajectory is a strict strong length minimizer.*

Remark 2. In Theorem 1, for Problem (Z) with a given submetric, the corresponding trajectory $\hat{w} = (\hat{z}, \hat{x}, \hat{u})$ can be nonsingular (i.e., not strictly abnormal).

Remark 3. In [1], Liu and Sussmann give examples of rigidity and minimality; these examples show that these properties do not imply each other in general. However, "nondegenerate" rigidity, namely, quadratic rigidity implies minimality by Theorem 1. Moreover, the following "converse" assertion holds: if a trajectory is strictly abnormal for a given submetric and satisfies quadratic sufficient conditions for weak minimality in the problem below, Problem (Z_*) (for these conditions, see [9–12, 15]), then it is quadratically rigid.

Theorem 1 is stronger than the result of Agrachev and Sarychev [6, Theorem 5.2] in which it is assumed that

a) $\dim \Gamma(x) = 2$,

b) the submetric is sub-Riemannian,

c) inequality (15) holds if we take a single $\psi \in G_a(\Psi_0)$ instead the whole set $G_a(\Psi_0)$ (which is a stronger assumption),

d) conditions in [6] provide a minimum with respect to the norm $\|w\|_1 = |z| + \|x\|_C + \|u\|_1$ instead of a strong minimum.

A point is said to provide a *Pontryagin minimum* if, for any N, it is a point of local minimum on the part of the admissible set defined by the additional condition $\|u\|_\infty \leqslant N$. This minimum condition is intermediate between those of the classical weak and strong minima. For any submetric, the last constraint obviously holds for some N, and hence the notion of Pontryagin minimum is equivalent to that with respect to $\|w\|_1$.

For 2-distributions, we can omit the strictness assumption for the hyperplane of support for the submetric in Theorem 1, but in this case we can guarantee only a Pontryagin minimum instead of strong minimum.

Theorem 2. *Suppose that* $\dim \Gamma(x) = 2$ *and that* $\hat{x}$ *is quadratically rigid. Then, for any submetric on* $\Gamma(x)$*, the trajectory* $\hat{w} = (\hat{z}, \hat{x}, \hat{u})$ *corresponding to an appropriate parametrization of* $\hat{x}$ (*such that* $\hat{z} = 1$ *and* $\varphi(\hat{x}(t), \hat{u}(t)) = 1$ *on* $[0,T]$) *is a strict minimizer of Problem* (Z) *with respect to the norm* $|z| + \|x\|_C + \|u\|_1$.

This theorem still implies Theorem 5.2 of [6]. For distributions of arbitrary dimension, this theorem holds under an additional assumption.

For any $a \in \mathbb{R}$, denote by $E_a(\Psi_0)$ the set of all $\psi \in G_a(\Psi_0)$ satisfying the relations [14, 15]

$$\psi(t)\,[\,[r_i, r_j], r_s]\,(\hat{x}(t)) \;=\; 0 \qquad \forall\, i, j, s = 1, \dots, k-1, \tag{16}$$

along $\hat{x}(t)$ on $[0,T]$. (In another form, these conditions were suggested in [10–13] and in earlier papers by the author.)

Definition 7. Following [7], we say that a trajectory $\hat{w}$ is *quadratically rigid in the Pontryagin sense* (or, briefly, Π-*quadratically rigid*) if $E_0(\Psi_0) \neq \varnothing$ and if there exists $a > 0$ such that

$$\Omega\,[E_0(\Psi_0)](\bar{w}) \;\geqslant\; a \int_0^T |\bar{y}|^2\, dt \qquad \forall\, \bar{w} \in \mathcal{K}. \tag{17}$$

This condition is equivalent [10] to the condition that the set $E_a(\Psi_0)$ with the same a is nonempty and

$$\Omega\,[E_a(\Psi_0)](\bar{w}) \;\geqslant\; a \int_0^T |\bar{y}|^2\, dt \qquad \forall\, \bar{w} \in \mathcal{K}. \tag{18}$$

This property again does not depend on the parametrization of $\hat{x}$, but now it *does depend* on the choice of the basis vector fields $r_1(x), \dots, r_{k-1}(x)$, or, more precisely, on the choice of a $(k-1)$-dimensional subspace $\Gamma_0(x) = \mathrm{Lin}\{r_1(x), \dots, r_{k-1}(x)\}$ transversal to $r_0(x)$. If $\psi \in G_a(\Psi_0)$, $a > 0$, and $k > 2$, then, for any x, relations (16) can hold for at most one subspace $\Gamma_0(x)$. For 2-distributions (i.e., for $k = 2$), there is only one relation in (16) in which $i = j = s = 1$, and it trivially holds in any basis, and hence Π-quadratic rigidity is equivalent to “ordinary” quadratic rigidity.

As was shown in [7], if $\hat{w}$ is Π-quadratically rigid, then it is Π-rigid, which means that, for any N, $\hat{w}$ is isolated with respect to the norm $\|w\|_1$ in the set of all trajectories of System (R) satisfying the uniform bound $\|u\|_\infty \leqslant N$.

Denote by Problem (Z_*) the problem obtained from Problem (Z) by replacing condition (2) by the condition $u_0 \leqslant 1$, i.e., the admissible control set becomes

$$U_* = \{\, u \mid \; u_0 \leqslant 1,\; u_1, \dots, u_{k-1} \text{ are free}\}.$$

If $\{u_0 = 1\}$ is a plane of support for $U(x)$ at $\hat{u} = (1, 0, \dots, 0)$, then we obviously have $U(x) \subset U_*$, and thus Problem (Z_*) is an extension of Problem (Z).

Theorem 3. *Let the trajectory* $\hat{w}$ *be* Π-*quadratically rigid in an adjoint basis. Then it is a point of strict Pontryagin minimum in Problem* (Z_*). *This means that, for any submetric such that*

$$\Gamma_0(x) = \mathrm{Lin}\{r_1(x), \dots, r_{k-1}(x)\}$$

is a plane of support at $r_0(x)$ *in a neighborhood of* $\hat{\chi}$, *the trajectory* $\hat{w}$ *is a strict minimizer in Problem* (Z) *with respect to the norm* $|z| + \|x\|_C + \|u\|_1$.

In particular, this theorem implies Theorem 2.

SUFFICIENT CONDITIONS FOR SMALL INTERVALS

Theorems 1–3 yield sufficient conditions for the minimality of the length for small intervals of the curve.

We again assume that $\hat{x}(t),\ t \in [0,T]$, is a Γ-admissible trajectory. Let $\hat{w} = (\hat{x},\ \hat{z} = 1,\ \hat{u} = (1,0,\dots,0)\,)$ be the corresponding trajectory of Problem (Z) and System (R). We consider these trajectories, as well as Problem (Z) and System (R), on the subintervals $\Delta = [t_1,t_2] \subset [0,T]$, and impose terminal relations of the form $x(t_1) = \hat{x}(t_1),\ x(t_2) = \hat{x}(t_2)$; we denote the corresponding objects by $\hat{x}_\Delta,\ \hat{w}_\Delta,\ (Z_\Delta)$ and (R_Δ), respectively.

Definition 8. According to [7], a trajectory $\hat{x}$ is said to be *quadratically rigid on small intervals* if there exists $\varepsilon > 0$ such that, for any $\Delta \subset [0,T]$, $|\Delta| \leqslant \varepsilon$, the set $G_+(\Psi_0(\hat{w}_\Delta))$ is nonempty, i.e., there exists a costate function $\psi(t)$ on Δ satisfying the adjoint equation (5), relations (4) and (11), and inequality (12) for some $a = a(\Delta) > 0$.

It is shown in [7, §6] that this property does not imply the stationarity of $\hat{w}$ on the entire interval [0,T]. On the other hand, this property implies the quadratic rigidity of $\hat{x}_\Delta$ for any sufficiently small interval Δ. (Note that, if $G_+(\Psi_0(\hat{w}_\Delta))$ is nonempty and Δ is sufficiently small, then (15) holds; see [7, 15].)

Definition 9. For a given submetric on $\Gamma(x)$, a trajectory $\hat{x}$ is said to be a *global* (*strict global*) *minimizer on small intervals* if there exists $\varepsilon > 0$ such that, for any $\Delta \subset [0,T]$, $|\Delta| \leqslant \varepsilon$, the trajectory $\hat{x}_\Delta$ is a global (strict global) length minimizer with respect to the given submetric, respectively.

Theorem 4 (Sufficient conditions for global minimality on small intervals). *Let $\hat{x}$ be quadratically rigid on small intervals. Then, for any submetric on $\Gamma(x)$ that has a strict hyperplane of support in a neighborhood of $\hat{x}(t)$, the trajectory $\hat{x}$ is a strict global minimizer on small intervals.*

This theorem generalizes results of Liu–Sussmann [1, Theorem 5] and of Agrachev–Sarychev [6, Corollary 5.2]. In both of these papers, it is assumed that

a) $\dim\Gamma(x) = 2$,

b) the submetric is sub-Riemannian,

c) the set $G_+(\Psi_0(\hat{w}\,[0,T]))$ is nonempty.

(Note that, in Theorem 4, it can happen that the trajectory $\hat{w}$ is not an extremal on the entire interval $[0,T]$, i.e., under our conditions, even the broader set $\Psi_0\,(\hat{w}\,[0,T])$ can be empty.)

For 2-distributions, in Theorem 4 one can omit the condition that the plane of support is strict. For $\Gamma(x)$ of arbitrary dimension, we again assume that the additional condition (16) holds. Set

$$E_+(\Psi_0(\hat{w}_\Delta)) \;=\; \bigcup_{a>0} E_a(\Psi_0(\hat{w}_\Delta)).$$

Definition 10. A trajectory $\hat{x}$ is said to be Π-*quadratically rigid on small intervals in an adjoint basis* if there exists an $\varepsilon > 0$ such that, for any $\Delta \subset [0,T]$, $|\Delta| \leqslant \varepsilon$, the set $E_+(\Psi_0(\hat{w}_\Delta))$ is nonempty, i.e., there exists a costate function $\psi(t)$ on Δ satisfying the adjoint equation (5), relations (4), (11), and (16), and inequality (12) for some $a = a(\Delta) > 0$.

(As above, for 2-distributions, any trajectory quadratically rigid on small intervals is automatically Π-quadratically rigid on small intervals in any adjoint basis.)

The following theorem is the analog of Theorem 3 for small intervals.

Theorem 5. *Let $\hat{x}$ be Π-quadratically rigid on small intervals in an adjoint basis. Then, for any submetric on $\Gamma(x)$ such that $\Gamma_0(x) = \mathrm{Lin}\{r_1(x),\dots,r_{k-1}(x)\}$ is a hyperplane of support at $r_0(x)$ in a neighborhood of $\hat{\chi}$, the trajectory $\hat{x}$ is a strict global length minimizer on small intervals.*

For the proofs of Theorems 1–5, see the forthcoming paper [16].

EXAMPLE

Consider a distribution $\Gamma(x)$ in $\mathbb{R}^3$ generated by the two following vector fields,

$$r_1(x) \;=\; \frac{\partial}{\partial x_1} \qquad \text{and} \qquad r_0(x) \;=\; b(x_1)\,\frac{\partial}{\partial x_2} \;+\; c(x_1)\,\frac{\partial}{\partial x_3}$$

with an arbitrary strictly convex submetric (in particular, an arbitrary inner product) on it. Here b and c are C^2 smooth functions satisfying the conditions

$$b(0) = 1, \qquad c(0) = c'(0) = 0, \qquad c''(0) \neq 0. \tag{19}$$

This example includes (as special cases) those of Montgomery [4], Liu–Sussmann [1, Sec. 2.3], and Petrov [3]. (In [4], $b \equiv 1$ and $c = x_1^2$; in [1], $b = 1 - x_1$ and $c = x_1^2$; in [3], $c(x_1) = b(x_1)x_1^2$ with a specific function b. All these papers consider sub-Riemannian metrics in which the basis (r_1, r_0) is orthonormal.)

The corresponding control system (2) is as follows: $\dot{x} = u_0\, r_0(x) + u_1\, r_1(x)$, i.e.,

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = b(x_1)\, u_0, \qquad \dot{x}_3 = c(x_1)\, u_0. \tag{20}$$

Consider the trajectory $\hat{x}(t) = (0, t, 0)$, $\hat{u}(t) = (1, 0)$, $t \in [0, T]$, where $T > 0$ is arbitrary. Then $\{r_0,\, r_1\}$ is an adjoint basis for $\hat{x}$, and System (R) becomes

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = z\, b(x_1), \quad \dot{x}_3 = z\, c(x_1), \quad \dot{z} = 0, \qquad x(0) = (0,0,0), \quad x(T) = (0, T, 0), \tag{21}$$

where the control u_1 is a free scalar, $\hat{u}_1 = 0$, and $\hat{z} = 1$. The set Ψ_0 consists of all normalized 3-vector functions $\psi(t)$ satisfying the adjoint equation (5), i.e.,

$$\dot{\psi}_1 = -\psi_2\, b'(0) - \psi_3\, c'(0), \qquad \dot{\psi}_2 = 0, \qquad \dot{\psi}_3 = 0, \tag{22}$$

and orthogonal to $\Gamma(x)$ along $\hat{x}(t)$,

$$\psi_1 u_1 + (\psi_2 b(0) + \psi_3 c(0))\, u_0 \;=\; 0 \qquad \forall\, u_0,\, u_1. \tag{23}$$

In view of (19), this readily implies that Ψ_0 consists of the two functions (in fact, constant vectors), $\psi = (0, 0, 1)$ and $\psi = -(0, 0, 1)$. For each of them, the Lagrange function is

$$\Phi\,[\psi](z, x, u) = \psi_3(0)x_3(0) - \psi_3(T)x_3(T) + \psi_3 \int_0^T (\dot{x}_3 - z\, c(x_1))\, dt,$$

and hence its second variation at $(\hat{z}, \hat{x}, \hat{u})$ is the functional

$$\Omega\,[\psi](\bar{z}, \bar{x}, \bar{u}) \;=\; -\psi_3 \int_0^T c''(0)\, \bar{x}_1^2\, dt. \tag{24}$$

We must consider this functional on the critical subspace, which, in particular, contains the equation $\dot{\bar{x}}_1 = \bar{u}_1$, $\bar{x}_1(0) = 0$. Since the Goh variable $\bar{y} = \bar{y}_1$ satisfies the same equation, $\dot{\bar{y}} = \bar{u}_1$, $\bar{y}(0) = 0$, we have $\bar{x}_1 = \bar{y}$, and, choosing $\psi_3 \;=\; -\operatorname{sign} c''(0)$, we obviously obtain

$$\Omega\,[\psi](\bar{z}, \bar{x}, \bar{u}) \;\geqslant\; |c''(0)| \int_0^T |\bar{y}|^2\, dt; \tag{25}$$

hence, by Theorem 1, the trajectory under consideration is a strict strong minimizer for any T and any strictly convex submetric. (For small T, by Theorem 4, it is a strict global minimizer.) This result is stronger than those obtained in [4, 1, 3]; in particular, in these papers, the minimality was proved for small T only.

Remark 4. Note that if $b'(0) = 0$, then it can happen that the above trajectory is not strictly abnormal for some submetrics. For instance, if $\varphi(x, u) = |u|$ in the above basis, then PMP is also satisfied with $\psi = (0, 1, 0)$, which does not belong to $\Psi_0(\hat{w})$ because it is not orthogonal to $\Gamma(\hat{x}(t))$. However, due to Theorem 1, this circumstance does not influence in the validity of the conclusion that $\hat{x}(t)$ is minimal.

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