M.S. Thesis

Criterion for the number of terms in the eigenfunction expansion method for acoustic inverse problems

Department of Mechanical Engineering

Major in Mechanical Engineering

The Graduate School, Yeungnam University

Nadezhda O. Kim

Advisor: Professor Lim Byoung-Duk

December, 2003

M.S. Thesis

Criterion for the number of terms in the eigenfunction expansion method for acoustic inverse problems

by

Nadezhda O. Kim

Advisor: Professor Lim Byoung-Duk

Thesis submitted to the Faculty of the Yeungnam University in partial fulfillment of the requirements for the degree of Master in Mechanical Engineering

December, 2003

Department of Mechanical Engineering
Major in Mechanical Engineering

The Graduate School, Yeungnam University

Approved as a qualified thesis of Nadezhda O. Kim
for the degree of Master in Mechanical Engineering
by the Evaluation Committee

Approved:

Chairman____________________

Kim Yong - Chul

____

Lee Won - Kyoung

____

Lim Byoung - Duk

December, 2003

The Graduate School, Yeungnam University

CONTENTS:

Nomenclature

p : Acoustic pressure distribution

U : Velocity distribution

$Pprec[\theta]$: Analytical solution of pressure

k : Wavenumber

ω : Natural frequency

c : Speed of sound

ρ : Density

ψ : Spherical functions

$h_n(kr)$: Hankel function

$P_n^m(Cos\theta)$: Legendre function of first kind

$Q_n^m(Cos\theta)$: Legendre function of second kind

$j_n(kr)$: Spherical Bessel of first kind

$y_n(kr)$: Spherical Bessel of second kind

λ : Wavelength

θ : Angle of spherical source

a : Radius of source

Mp : Number of microphones

SNR : Signal to noise ratio

p_m : Measurement acoustic pressure signals

r, θ, ϕ : Spherical coordinates

List of figures

I. INTRODUCTION

In engineering applications it is often required to tackle problems of noise radiation from machines that already built. A common approach is to identify the noise surface and radiation paths so as to block the transmission. However, this is not easy to do because sound can not be seen. For engineers to diagnose noise sources and block sound transmission, it would be highly desirable to develop an effective tool to visualize the radiated acoustic pressure field from a vibrating source.

The objectives of the present work are: to derive wavenumber vector for spherical wave; to relate spatially varying wavenumber to maximum resolution of eigenfunctions; to set a criterion for the highest order of expansion functions, which depends on the resolution of measurement.

The radiated acoustics pressure signals are were taken as analytical solution of vibrating spherical cap and used as an input to the HELS method to reconstruct the acoustic pressure fields. The reconstructed acoustic pressures and spectra are then compared to those simulated data at the same locations.

To determine acoustic pressures on the discrete nodes on the source surface, one must take the same number of measurements in the field. Since these numbers increase with the excitation frequency and complexity of source geometry, the resulting diagnostic process may be extremely time-consuming and costly.

The HELS method uses expansion functions to represent the radiated acoustic pressure field, and the number of measurements is determined by that of expansion functions.

The HELS method enables one to reconstruct the radiated acoustic pressures over the entire surface of the source based on measurements taken over a finite measurements surface. This capability could make the HELS method potentially a viable noise diagnostic tool.

In this research reconstruction of the acoustic field is done by directly solving the Helmholtz equation (Wang 1995). The acoustic is expanding in terms of set independent functions generated by Gram-Schmidt orthonormalization with respect to the particular solutions to the Helmholtz equation on the particular vibrating surface under consideration. The coefficients associated with these independent functions are determined by requiring the assumed from of the solution to satisfy pressure boundary conditions at the measurement points. The errors incurred in this process are minimized by the least-square method. Once these coefficients are specified, the acoustic pressure anywhere, including the source surface, is completely determined. Since the number of expansion terms determines that of measurements, which is usually much smaller than that of discretized nodes on the surface, reconstruction of the acoustic pressure field can be done efficiently. It also depends on optimal number of expansions, which we want to determine. Moreover, solutions thus obtained are always unique.

II. THEORETICAL BACKGROUND

What acoustic inverse problem is?

1. Input data are pressure, waves, sound and excitation force.
2. Reconstruct acoustic pressure (velocity distribution on the source) using eigenfunction expansion method.
3. Obtain properties of the source.

1.1 Boundary value problems

The acoustic pressure radiated from a vibrating object into an unbounded fluid medium satisfies the wave equation, whose Fourier transformation is known as the reduced wave equation or Helmholtz equation.

$$\nabla^2 p + k^2 p = 0 \qquad (1)$$

where p represents the complex amplitude of the acoustic pressure and $k = \frac{\omega}{c}$ is the wave number. At the interface with the surface of the vibrating object, p satisfies one of three types, or their combination of boundary conditions.

Dirichlet problem: $p(x_B) = g(x_B)$

Neumann problem: $\frac{\partial p(x_B)}{\partial n} = g(x_B)$ (2)

Mixed or Robin problem: $a(x_B)p(x_B) + b(x_B)\frac{\partial p(x_B)}{\partial n}$

Where $x_B \in \partial B, a, b, g$ are specified, and $\frac{\partial}{\partial n}$ represents the normal derivatives on the boundary ∂B.

Solutions to equation (1) subject to boundary conditions (2) can be approximate by a linear combination of the independent functions ψ^*

$$p^* = \rho c \sum_{i=1}^{n} C_i \psi_i^* \qquad (3)$$

where ρ and c are density and speed of sound of the fluid medium, respectively. The independent functions ψ_i^* can be selected in such a way that they satisfy any one of the following three conditions:

(a) ψ_i^* satisfy the differential equation, but not the boundary condition;

(b) ψ_i^* don't satisfy the differential equation, but ψ_i^* satisfies the boundary conditions and

$\psi_2^*, \psi_3^*, \ldots$ satisfy the homogeneous boundary conditions; or

(c) ψ_i^* satisfy neither the differential equation nor the boundary conditions.

In this research, we focus our attention on the Helmholtz equation (1) subject on the boundary (2), and select the independent functions ψ_i^* that satisfy the condition (a).

1.2 Orthonormalization of the spherical functions ψ_i^*

The independent functions ψ_i^* in equation (3) can be derived from the spherical wave functions through an orthonormalization process. To this end, let us rewrite equation (3) in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial p}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial p}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial p}{\partial \phi^2} + k^2 p = 0 \qquad (4)$$

Subject to Dirichlet boundary condition, equation (4) and Summerfield's radiation condition at infinity,

$$\lim_{r->\infty}[r(\frac{\partial p}{\partial r} - ikp)] = 0 \qquad (5)$$

Physically, it is means that we didn't expect traveling waves from infinity. With respect to equation (4) can be written in the form of equation (3), with ψ_i^* being generated by a linear combination of the particular solutions to the Helmholtz equation ψ_i

$$\psi_{m,n}(r,\theta,\varphi) = h_m(kr)P_{n,m}(\cos\theta)\begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix} \qquad (6)$$

Where, $h_m(kr)$ - Hankel function and $P_{m,n}(\cos\theta)$ - Legendre polynomial.

This functions ψ_i given by Eq.(6) are readily applicable to a spherical source, but may be not ideal for irregularly shaped geometries, especially for those that contain sharp edges. However, in engineering applications true sharp edges are rare (uncommon). They are often rounded. Moreover, radiated acoustic pressure from a finite, irregularly shaped source obeys the spherical spreading law in the far field. Hence equation (6) may be used as a first approximation.

The independent functions ψ_i^* can be generated by the Gram-Schmidt orthonormalization with respect to ψ_i

$$X_1 = \psi_1 \qquad \psi_1^* = \frac{X_1}{\|X_1\|}$$
$$X_2 = \psi_2 - (\psi_2, \psi_1^*)\psi_1^* \qquad \psi_2^* = \frac{X}{\|X_2\|}$$
$$\ldots \qquad \ldots \tag{7}$$
$$X_{n+1} = \psi_{n+1} - \sum_{i=1}^{N}(\psi_{n+1}, \psi_i^*) \qquad \psi_{n+1}^* = \frac{X_{n+1}}{\|X_{n+1}\|}$$

$\langle \psi_i(x_B,\omega), \psi(x_B,\omega) \rangle = \int_{\partial B} \psi_i(x_B,\omega)\psi_j(x_B,\omega)dS$ - Inner product on the source surface.

The independent functions ψ_i^* thus obtained on the particular source boundary ∂B.

For each source we should take inner product individually.

1.3 The Helmholtz equation-least-squares method

The HELS method expresses the radiated acoustic pressure in terms of an expansion of propagating wave functions.

$\hat{p}(x)$-complex amplitude of the radiated acoustic pressure in any field point x, ρc is the characteristic impedance of the fluid medium.

Consider a close, smooth, and impermeable surface ∂B immersed in and unbounded fluid medium B. Assume that the surface vibrates at a constant angular frequency ω. There are no other sources in the medium except the vibrating surface under consideration.

Given the source location, geometric shape, and pressure boundary conditions at the measurement points in the field $p(x_S) = p_0(x_S)$ - (8) we wish to reconstruct the acoustic pressure on the source surface.

Equation (3), here ψ_i^* comes from equation (6) through orthonormalization (7).

The inner products in the orthonormalization process are taken over the entire source boundary ∂B. Then the coefficients C_i are determined by requiring assumed form of solution (3), to satisfy the boundary conditions (8). Suppose that N-term expansion in equation (3) is used, and M measurements are taken, where $M \geq N$. Next, we can form M-simultaneous algebraic equations for N unknowns

Note that there are no restrictions on the locations of measurements so long as they do not overlap. If an N-term expansion in Eq. $\rho c \sum_{n=1}^{N} C_i \psi_{mi}^*(x_m) = p_m^*(x_m)$ is used, then M measurements are taken.

Theoretically, if the measured acoustic pressure $p_m^*(x_m)$ is exact, then the assumed-form solution converges to the true value as $N \to \infty$. However this never happens in reality because the measured quantities $p_m^*(x_m)$ always contain errors due either to measurement uncertainties or to rapid decay of the near-field effects. While the former can be reduced by taken more averages in the measurements, the later is irreversible. Moreover, the number of measurements M is always finite. Hence the reconstructed acoustic pressure will never converge to the true value.

$$\begin{bmatrix} \psi_{11}^* & \dots & \psi_{1N}^* \\ \dots & \dots & \dots \\ \psi_{M1}^* & \dots & \psi_{MN}^* \end{bmatrix} \begin{Bmatrix} C_1 \\ \dots \\ C_N \end{Bmatrix} \approx \begin{Bmatrix} p_{01} \\ \dots \\ p_{0N} \end{Bmatrix} = \begin{Bmatrix} p_1(x_m) + n(x_m) \\ \dots \\ p_1(x_m) + n(x_m) \end{Bmatrix} \quad (9)$$

$n(x_m)$ - Noise

The matrixes are not squares; therefore we cannot solve this system directly.

If the measured quantities p_0 are exact, then the approximate solution p^* converges to the true value as $N \to \infty$. However, indeed, the measurements always contain certain uncertainties due either to random fluctuations or to the effect of rapid decay of the evanescent waves (Non- propagating mode).

Hence, the approximation solution will not converge to the true one. Nevertheless, the error involved in the approximation solution can be minimized by the least-squares method.

$I = \sum_{m=1}^{M} [\rho c \sum_{n=1}^{N} C_n \psi^*_{mn} - p_{om}]^2$ - Criterion function (10)

Here C_n are variables.

Goal is to find optimal number N of expansion terms, in order to minimize error.

And we will find extreme of this function:

$$\frac{\partial I}{\partial C_i} = \frac{\partial}{\partial C_i}[\sum_{m=1}^{M}(\rho c \sum_{n=1}^{N} C_n \psi^*_{mn} - p_{0m})^2] = \sum_{m=1}^{M} 2(\rho c \sum_{n=1}^{N} C_n \psi^*_{mn} - p_{0m}) \times \frac{\partial}{\partial C_i}(\rho c \sum_{n=1}^{N} C_n \psi^*_{mn}) =$$

$$\left| \frac{\partial C_n}{\partial C_i} = \delta_{ni}, \delta_{ni}\psi^*_{mn} = \sum \psi^*_{mi} \right| = 2(\rho c \sum_{m=1}^{M} \psi^*_{mi} \sum_{n=1}^{N} C_n \psi^*_{mn} - \sum_{m=1}^{M} p_{0m}\psi^*_{mi}) = 0$$

(11) $i = 1,2,\ldots,N$

From here we obtain coefficients $C_{\min}$ - extreme

To enhance the accuracy of reconstruction Eq (10) can be written in matrix form $[\Gamma]\{C\} = \{B\}$ (12)

Where $[\Gamma]$ - represents the transformation matrix that correlates the field measurements to the acoustic field on the source surface, and $\{B\}$ contains the information of the boundary condition.

$\Gamma_{mi} = \rho c \sum_{n=1}^{N} \psi^*_{nm}(x)\psi^*_{ni}(x)$ and $B_m = \sum_{n=1}^{N} p_{0n}(x)\psi^*_{nm}(x)$ (13)

Transformation matrix is nonsingular; therefore the coefficients C_i can be solved by inverting the matrix $[\Gamma]$

$\{C\} = (\rho c)^{-1}[\Gamma\ ]^{\mu}\{p_0\}$ (14)

$[\Gamma]^{\mu} = ([\psi^*_{mn}]^T[\psi^*_{mn}])^{-1}[\psi^*_{mn}]^T$ Pseudoinverse matrix

Here $[\psi^*_{mn}]$ represents a $M \times N$ matrix that consist the expansion functions $\psi^*_{mn}(x)$. It can be shown that the condition number of the pseudoinverse matrix $[\Gamma]^{\mu}$ much smaller than that of direct inverse. Hence the accuracy of numerical computations is high.

In deriving Eq. (14) no restriction have been imposed on the measurement points. They can be taken at any points in the field as long as they do not overlap each other.

Once the coefficients C_i are solved, the surface acoustic pressures can be reconstructed by equation (3). In fact, one can use equation (3) to reconstruct acoustic pressure anywhere external to the vibrating surface.

The basis functions are expressed in terms of the spherical coordinates. This because the spherical Hankel functions and Legendre functions are readily available in many mathematical libraries, such as

IMSL (International Mathematical and Statistical Library) subroutines. Use of these functions can be extremely effective for spherical sources. For an elongated and flat object, the prolate, oblate, or elliptic coordinates may be used, respectively, to provide faster convergence in numerical computations. However, the spherical functions in these coordinates are not available in many mathematical libraries, and direct numerical computations of these functions can be time consuming.

$$\psi_{m,n}(r,\theta,\varphi) = h_m(kr)P_{n,m}(\cos\theta)\begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}$$

Spherical Hankel function:

$$h_n(kr) = j_n(kr) + y_n(kr)$$

Spherical Bessel functions, first and second kind

$j_n(z) = \frac{1}{\sqrt{z}} J_{n+\frac{1}{2}}(z)$ and $y_n(z) = \frac{1}{\sqrt{z}} Y_{n+\frac{1}{2}}(z)$

[A. Korn, M. Korn “Mathematical handbook for scientists and engineers”]

It is emphasized that the HELS method is free of the nonuniqueness difficulty, because it solves the Helmholtz equation directly. Hence the reconstructed surface acoustic pressures are always unique, even at the characteristic frequencies that correspond to the related interior boundary value problem.

(15) $\vec{u} = \frac{\nabla p}{j\omega\rho} = \frac{1}{j\omega\rho}(\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial p}{\partial \theta})$ Assumed velocity distribution.

Compare velocity distribution, simulated numerically, and assumed velocity distribution. Find out how many number of terms do we need for eigenfunction expansion or how many measurement do we need

take, in order to minimize error of reconstruction or difference between assumed velocity distribution (by HELS) and generated velocity distribution from source surface (numerically).

In all cases, a surface velocity distribution is preselected, and the acoustic pressures on the source surface and in the field are calculated either analytically or numerically. The field acoustic pressure thus obtained are taken as the input to reconstruct the surface acoustic pressures, which are subsequently compared with the predetermined ones on the surface.

The expansion functions ψ_j are frequency dependent; the associated coefficients C_j must be solved repeatedly for each frequency.

[Zaoxi Wang and Sean F. Wu (1997) "Helmholtz equation-least-squares method for reconstructing the acoustic pressure field"]

III. ADVANTAGE AND LIMITATION OF THE HELS METHOD

Important advantage of the HELS method is its simplicity in expressing the acoustic pressure in terms of spherical functions, which are the basis functions. The basis functions we use as a superposition of multipole expansion, with its coefficients, which was determined by matching the assumed-form solution with the pressure boundary condition at the measurement points.

As other methods, the HELS method has limitations. Theoretically, the HELS method may yield an exact reconstruction of radiated

acoustic pressure field from a spheroid such as a spherical, oblate, prolate, elliptic spheroids, etc. Among these spheroids, the spherical surface is the most convenient because a basis functions are available and numerical computation can be extremely efficient.

For a nonspheroid surface the acoustic pressure field reconstructed by the HELS method may be valid outside a minimum spheroid that encloses the source but is approximate at best inside this control surface.

Mark out one can us the multipole expansion theory to reconstruct the acoustic pressure field with its coefficients determined by the orthogonality properties of expansion functions. Here, the reconstructed acoustic pressure would be exact outside the minimum sphere that encloses the source, but invalid inside the control surface. The HELS uses a superposition of multipole expansion with its coefficients determined by forcing the assumed –form solution to satisfy the pressure boundary condition at measurement locations.

Obviously, the accuracy of reconstruction depends on the accuracy of the input data, which is that case for all inverse problems. If the problem under consideration is to reconstruct the far-field acoustic pressure, then the accuracy of reconstruction is determined primarily by the measurement uncertainties. The error due to the lost of a near-field acoustics pressures in the input will have a negligible impact on reconstruction, because this near-field effect decays exponentially with distance. On the other hand, if acoustic pressures on the source surface are sought, as in the case of noise diagnostics, the accuracy of reconstruction will depends critically on that on the measured near-field acoustic pressures. If this information is incomplete, then there will be

no way of reconstructing accurately the surface acoustic pressure which is predominated by the near-field effect. Under these circumstances, increasing the number of expansion terms N will not help; indeed, it may even worsen the accuracy of reconstruction.

This is because the problem is already ill-posed; hence adding more terms may simply introduce more errors and eventually cause the solution to diverge beyond a certain value of N.

It is important that in engineering application the acoustic pressure near field cannot be captured completely. Therefore an exact reconstruction of the radiated acoustic pressure distribution over the surface of a complex vibrating structure is not possible. Taking more measurements at closer distances can improve the accuracy of the near-field acoustic pressure measurements, thus leading to a more accurate reconstruction. However, this may prolong the diagnostic process and even make it impractical.

[Sean Wu (2000) “On reconstruction of acoustic pressure fields using the Helmholtz equation least square method”]

IV. IMPLEMENTATION OF THE HELS METHOD

The goal is to develop a simple yet effective methodology to reconstruct the radiated acoustic pressure fields. Therefore we must devise a procedure that can yield the desired accuracy with relatively few measurements. The questions that must be addressed include:

(1) Where and how the measurements should be taken? (2) How many measurements are necessary to achieve the desired solution? and (3)

Given the number of measurements, what is the optimal number of expansion terms?

Unfortunately, there are no definite answers to all these questions because of the uncertainties involved in an inverse problem. What we can do is to develop some guidelines with which satisfactory reconstruction of the radiated acoustic pressure fields can be obtained.

Numerical results have demonstrated that in general the closer the measurement to a source surface, the more accurate the reconstructed acoustic pressure. Therefore we must try to take measurements in the near field. Assume that the measurement distance is d, the characteristic dimension of a source - a, and the maximum frequency of interest $f_{\max}$. And the value of d should satisfy the following three conditions simultaneously: (1) $d \ll a$, (2) $d \ll \lambda_{\min} = \frac{c}{f_{\max}}$, and

(3) $d \ll k_{\max} \frac{a^2}{2}$, where $k_{\max} = \frac{2\pi}{\lambda_{\min}}$. Also, it would be better to take measurements over a conformal surface that completely encloses the source.

The method was based on spherical wave expansion theory to reconstruct acoustic pressure field from a vibrating object. The HELS method was used to reconstruct acoustic pressure distribution and velocity distribution. Basically, HELS method is experimental method; it needs measurement data or numerical simulation data. However, analytical solution was used as an input data, therefore, it is ideal case. It means that method works best, if result of HELS method is coincide with analytical solution.

The optimal number of expansion terms seems to be reasonable. Physically, the high- order terms shows the near-field effect, which may have been lost in the measurement data and can not be recovered anyway. Also, the low-order terms depict the propagating wave effect, which may be captured in the measured data and, therefore, can be reconstructed. Consequently, by using an optimal number of basis functions it is possible to obtain a satisfactory reconstruction of the radiated acoustic pressure field in near and far fields in a cost-effective manner.

["Helmholtz equation-least-squares method for reconstructing the acoustic pressure field" Zaoxi Wang and Sean F. Wu]

V. ANALYSIS OF SPHERICAL FUNCTIONS

5.1 Definition of wavenumber

$\nabla^2 p = \dfrac{1}{c^2}\dfrac{\partial^2 p}{\partial t^2}$ Wave equation

When solution $p(x,y,z,t) = A(x,y,z)e^{j\omega[-\Gamma(x,y,z)/c_0]}$ has substituted in wave equation:

$$\frac{\nabla^2 A}{A} - \left(\frac{\omega}{c_0}\right)^2 \nabla\Gamma \cdot \nabla\Gamma + \left(\frac{\omega}{c_0}\right) = 0$$

$$\frac{\omega\Gamma}{c_0} = \phi\,, \quad \nabla\phi \cdot \nabla\phi = \frac{\nabla^2 A}{A} + k^2$$

$$A(r,\theta,\phi) = h_n(kr)(P_n^m(Cos\theta) + iQ_n^m(Cos\theta))$$

$$\nabla^2 A = A1 + A2 = \frac{\partial}{\partial r}\left(r^2 \frac{\partial A}{\partial r}\right) + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial\theta}\left(Sin\theta \frac{\partial A}{\partial\theta}\right)$$

Legendre functions of first and second are eigenfunction of operators

$$\left|\frac{\nabla^2 A}{A}\right| \ll 1 \qquad \Rightarrow \quad k = \nabla \phi$$

$$\nabla \phi = \nabla[h_n[kr](P_n^m[Cos\theta] + iQ_n^m[Cos\theta])e^{-j\omega\phi}]$$

m=0 axisymmetrical case

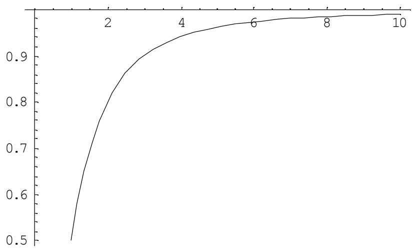
5.2 Component of wave number with respect to r

$$k_r[r,n] = \frac{\partial[\tan^{-1}[\frac{y_n[kr]}{j_n[kr]}]]}{\partial r}$$

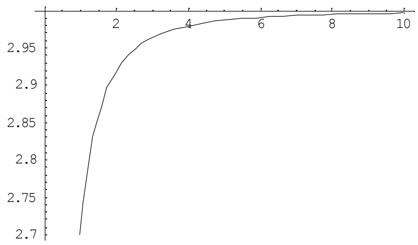
Wavenumber component depends on n & m

n=1

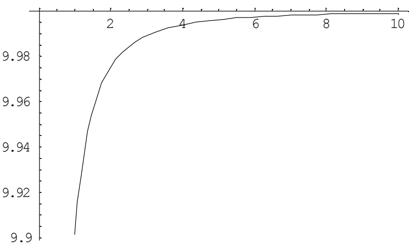
k=1



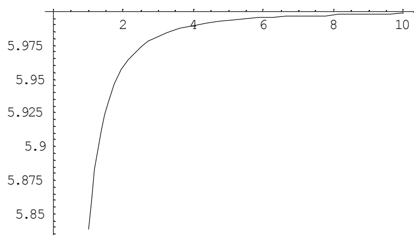
k=3



k=10

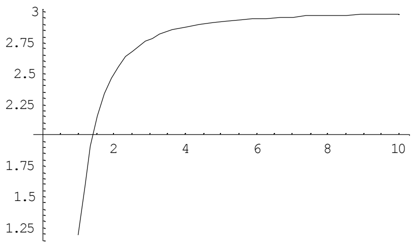


k=6

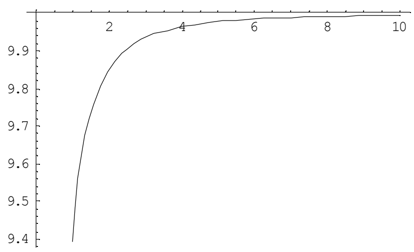


n=3

k=3

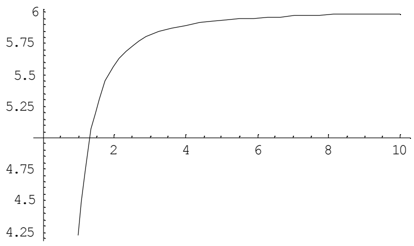


k=10



n=4

k=6



k=7

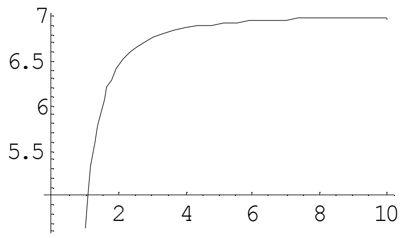
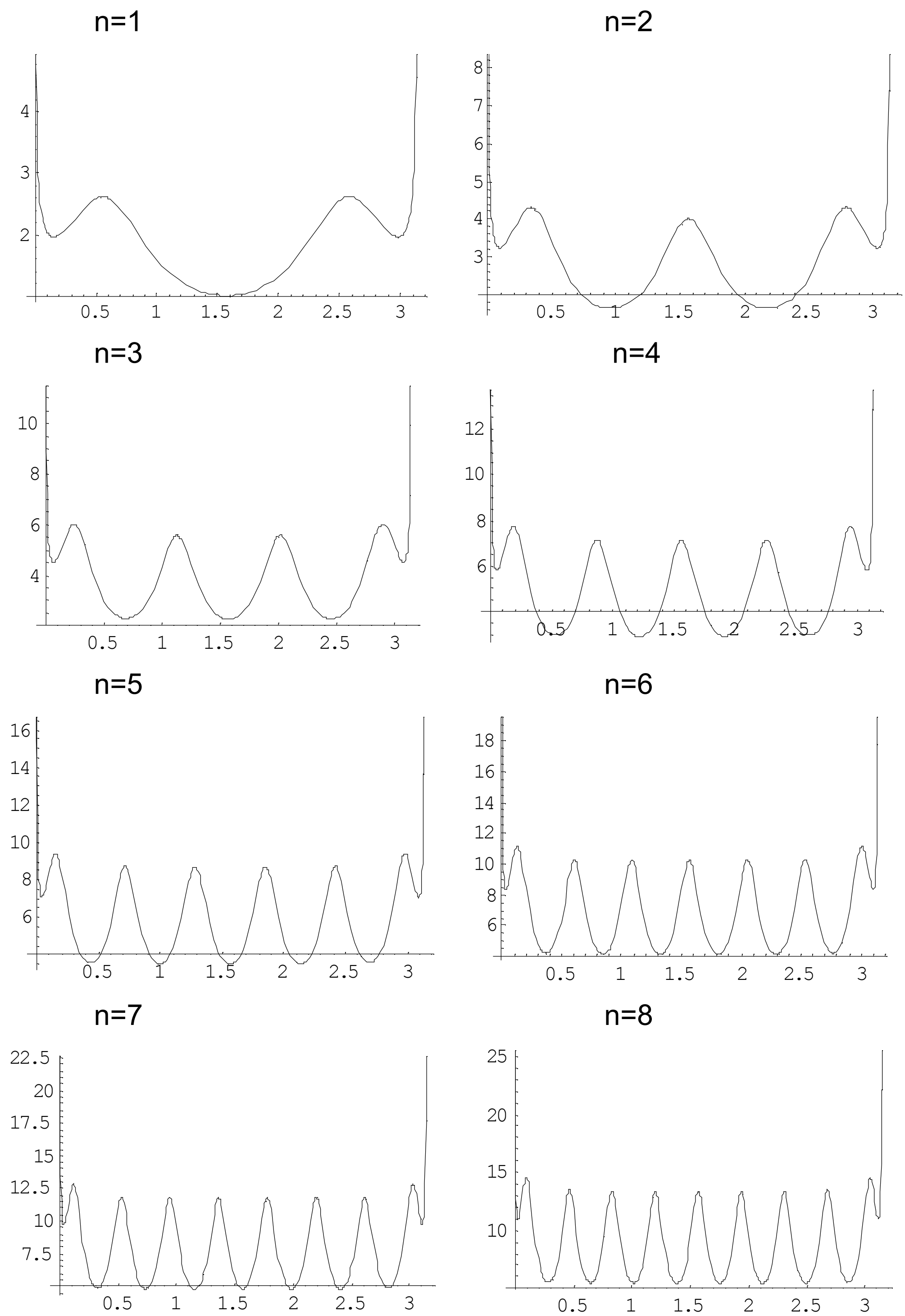


Fig.1 Wavenumber component k_r with respect to radius

The component of wave number with respect to r converges to the value of wave number.

5.3 Component of wave number with respect to θ

$$k_\theta[\theta, n, r] = \frac{1}{Sin[\theta](P^2[n,m,Cos[\theta]] + Q^2[n,m,Cos[\theta]])}$$


n=1
n=2
n=3
n=4
n=5
n=6
n=7
n=8

n=9

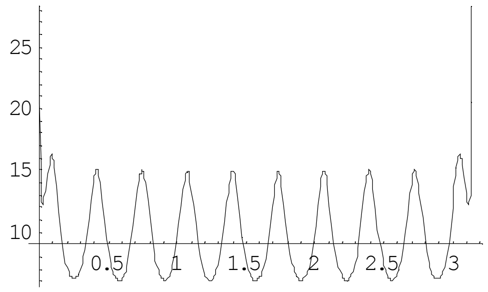


Fig.2 Wavenumber component k_θ with respect to θ

Component θ wave number has wave behavior, namely, the number of Legendre dimensional equal to the number of deeps.

Component $rk_\theta = \dfrac{1}{Sin[\theta](P^2[n,m,Cos[\theta]] + Q^2[n,m,Cos[\theta]])}$ n=4, m=2

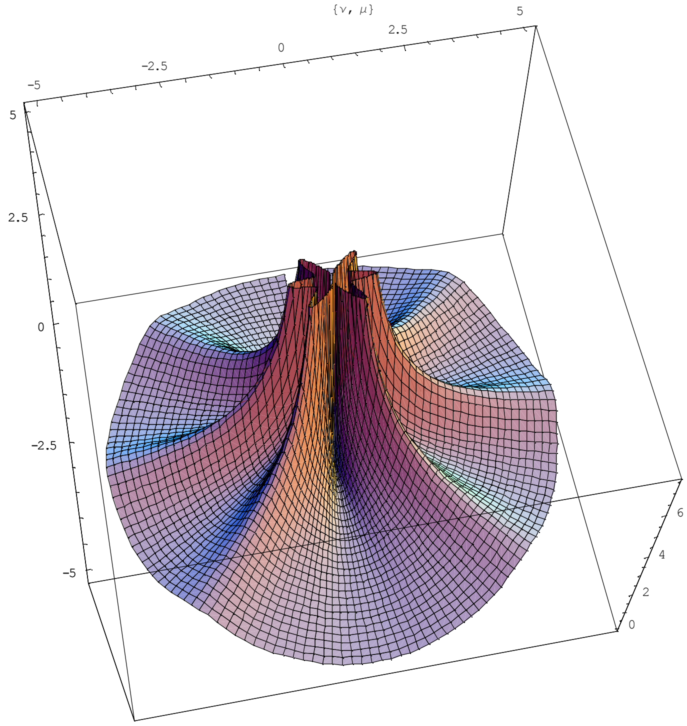


Fig.3 Wavenumber component in 3 DOF

5.4 General solution or pressure distribution:

$$p(r,\theta,\phi) = \sum_m \sum_n A_{mn} h_n(kr)(P_n^m(cos\theta) + iQ_n^m(\cos\theta))e^{im\phi}$$

The eigenfunction is $\psi_{mn}(\theta,r,\phi) = h_n(kr)(P_n^m(\cos\theta) + iQ_n^m(\cos\theta))\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}$

Wave number is $\nabla\phi\cdot\nabla\phi = \frac{\nabla^2 A}{A} + k^2$, $\left|\frac{\nabla^2 A}{A}\right| \ll 1$

$|\nabla f|^2 \sim k^2$

k_r is real – propagating wave, it becomes extinct as $\frac{1}{\sqrt{r}}$.

k_r is imaginary – wave doesn't propagate, namely it becomes extinct very fast as $\exp(-z)$.

$h_n(ikr) \sim BesselK_n(z) \sim \exp(-z)$

$z \to 0 \quad h_n(ikr) \to +\infty$

$z \to -\infty \quad h_n(ikr) \to 0$

The magnitude of amplitude k_θ is increase as n goes to infinity. This dependence is linear, the graph is sort of $y(x) = kx$.

Coefficient k approximately equals $\frac{\pi}{2}$ and graph monotonically increases.

linear equation for k_θ dependence on $n \;\to\; 0.8 + \frac{\pi}{2}n$

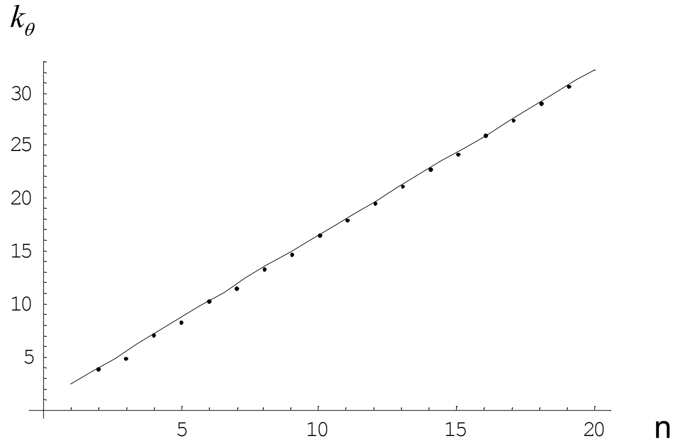


Fig.4 Wavenumber component k_θ with respect to n

$$rk_\theta \sim \frac{\pi}{2} n$$

$$(k_\theta)^2_{nm} + m^2 = (k_\theta)^2_{n0}$$

$k_r(n) \sim (k_\theta)_{n0}$ Inequality relationship

Wave number component depending on m

n verify - {4,7,10,15,20} and m is variable

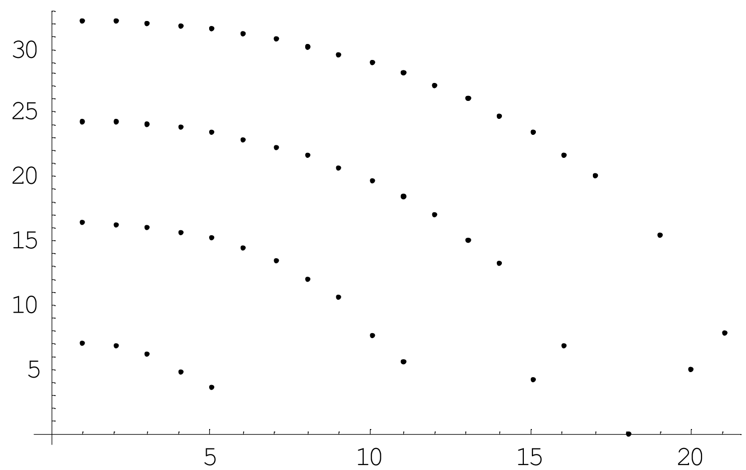


Fig.5 Wavenumber component k_θ with respect to even m

This is criterion for n and m, which should be included in the series solution to a specific radiation (or inverse) problem.

$$rk_{\theta} \sim \frac{\pi}{2} n$$

$$rk_{\theta} = \frac{\pi}{2}\sqrt{(n+\frac{1}{2})^2 - m^2} = \frac{\pi}{2}\sqrt{(n+\frac{1}{2})^2 - k_{\phi}^2}$$

VI. GENERAL SOLUTION

Returning to the separation of variables, we combine the solution back together. There are six possible combinations with the two independent solutions for each coordinate.

$$p(r,\theta,z,t) \propto H_n^{(1),(2)}(k_r r) e^{\pm in\phi} e^{\pm ik_z z} e^{-i\omega t}$$

The most general solution in frequency domain is given by:

$$p(r,\phi,z,\omega) = \sum_{n=-\infty}^{+\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{+\infty} [A_n(k_z,\omega) e^{ik_z z} H_n^{(1)}(k_r r) + B_n(k_z,\omega) e^{ik_z z} H_n^{(2)}(k_r r)] dk_z$$

VII. SPHERICAL WAVE SPECTRUM

Spherical wave spectrum $P_{mn}(r)$ at $r = r_0$ as

$P_{mn}(r_0) = \int p(r_0,\theta,\phi) Y_n^m(\theta,\phi)^* d\Omega$ (1), which decomposes the pressure on the sphere of radius r_0 into its spherical wave components.

For the sphere the wave is composed of a traveling wave component in ϕ using $e^{im\phi}$ and standing wave component in θ given

by $P_n^m(Cos\theta)$. Because of the standing wave component we can't define wave fronts, as we did with the plane wave spectra. One could expand the Legendre functions into traveling wave components using $Cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ to develop expressions purely in terms of traveling waves $(e^{im\phi + in\theta})$.

As was done for planar and cylindrical systems we can view

$$P_{mn}(r_0) \equiv \int p(r_0, \theta, \phi) Y_n^m(\theta, \phi)^* d\Omega$$

as a forward Fourier transform using $Y_n^m(\theta, \phi)$ as the basis functions. The corresponding inverse Fourier transform (a double Fourier series) is just

$$p(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} P_{mn}(r) Y_n^m(\theta, \phi) \qquad (2)$$

due to the orthogonality of the basis functions.

Comparison ith $p(r, \theta, \phi) = \sum_{n=0}^{\infty} \frac{h_n(kr)}{h_n(ka)} \sum_{m=-n}^{n} Y_n^m(\theta, \phi) \int p(a, \theta', \phi') Y_n^m(\theta', \phi')^* d\Omega'$,

where $d\Omega' = Sin\theta' d\theta' d\phi'$ reveals that $P_{mn}(r) = \frac{h_n(kr)}{h_n(kr_0)} P_{mn}(r_0)$, which provides wave field extrapolation very similar to the cylindrical wave case given in eq.(3)

$$P_n(r, k_z) = \frac{H_n^{(1)}(k_r r)}{H_n^{(1)}(k_r a)} P_n(a, k_z) \qquad (3)$$

When $r \geq r_0$ the extrapolation is a forward one, and when $r < r_0$ it is an inverse one, equivalent in the time domain to going back in time.

VIII. EVANESCENT WAVE

In consideration of the result from the study of plane waves we would expect evanescent waves to be generated when the acoustic wavelength is larger than the wavelength in the axial and / or circumferential direction. (For cylindrical wave)

However, unlike the plane wave case, there is a difference between the axial and circumferential cases. Developer is giving rise to true exponential decay and the latter to a power low decay.

We will look at both cases. The wavelength in the circumferential direction is given by $\lambda_\phi = \dfrac{2\pi a}{n}$, where $2\pi a$ is the circumference and n is the number of complete cycle in the circumference.

First consider the axial case. When the wavelength in the axial direction is smaller than the acoustic wavelength λ (λ_{medium}) then one would expect a decay of energy from the surface at $r = a$. These decaying, non-propagating waves are called subsonic or evanescent waves and exhibit an exponential decay away from the surface. That is, when $\lambda_\phi \to \lambda_{madium}$

The wave number in spherical coordinate system decrease as we move to the poles.

$k > k_{medium}$ on the equator and $k \to k_{medium}$ on the poles.

$$P_n(r, k_\phi) = \frac{h_n^{(1)}(k\,r)}{h_n^{(1)}(k\,a)} P_m(a, k_\phi) \qquad (4)$$

One can show that the radial velocity for this wave is in phase quadrature with the pressure so that no energy is carried away from the shell by this wave.

And now consider evanescent conditions in the circumferential direction which occur when the circumferential wavelength $\lambda_\phi < \lambda$. Assume that $k_\phi < k_{medium}$ and k is real, that means that the axial wave is supersonic. For example, set $k_\phi = 0$ (infinite axial wavelength) and note that Eq.(4) applies, that is, the Hankel functions of real argument govern the decay. When $r \gg n$ the ratio of Hankel functions approaches $\dfrac{h_n^{(1)}(k_r r)}{h_n^{(1)}(k_r a)} \approx \sqrt{\dfrac{a}{r}} e^{ik_r(r-a)}$, and field decay as expected for spherical wave, proportional to the square root of the radial distance. There is no evanescent behavior here. However, since $\lambda_\phi < \lambda$ one anticipates some kind of short circuit of the radiation of this wave from the surface $r = a$, since the medium only supports radiation at the characteristic wavelength λ as implied by the Helmholtz wave equation. This short circuit should becomes more complete as the index of the Hankel function n becomes larger since n is the number of wavelength.

The short circuit can be demonstrated mathematically by keeping the argument of Hankel functions fixed and allowing the order to increase so that we can use asymptotic expansions for large orders. The asymptotic expansion as $n \to \infty$ for Hankel function is

$$h_n(\zeta) = \frac{1}{\sqrt{2\pi n}}\left(\frac{e\zeta}{2n}\right)^n - i\sqrt{\frac{2}{\pi n}}\left(\frac{e\zeta}{\pi n}\right)^{-n}, \qquad (5)$$

where $\zeta = kr$ .When $\frac{\zeta}{n} < 1$ then we can ignore the real part of Eq.(7) we find that the n-th component of the pressure P_n becomes

$$P_n(r,0) \approx (\frac{a}{r})^n P_n(a,0) \qquad (6)$$

This equation works when $kr < n$, which is equivalent to the evanescent wave condition $\frac{2\pi r}{\lambda} < n$.

Fig.6 demonstrates the power law decay. Here the exact values of the ratio of Hankel functions are plotted as a function of $Abs[\frac{h_n(kr)}{h_n(ka)}]$, $a = 1$ (radius of sphere) for 8 different values of n . Note that these lines separate the differing slope regions on each curve. To the right of these lines the wave is spherically spreading, and to the left it is evanescent.

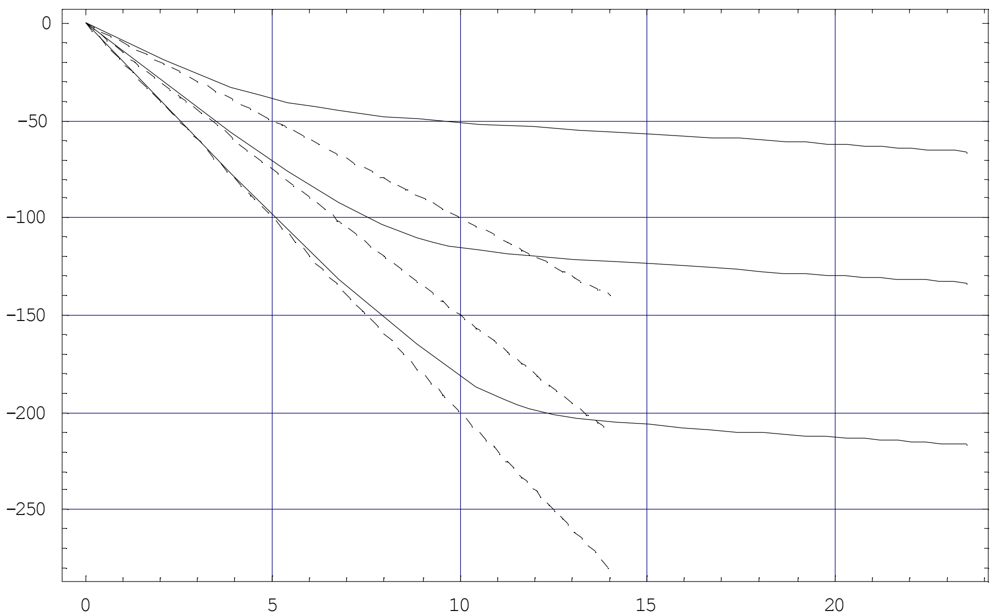


Fig.6 dB ratio of Hankel functions when ka=5 and $k_z = 0$

The n is the wave number of the wavelength, which fit around the circumference of the wave front at the radius r. Therefore whenever the number of wavelengths is less than n, P_n will decay inversely with the n-th power of the distance. This is the sought-after evanescent

$$P_{mn}(r) = \frac{h_n(kr)}{h_n(ka)} P_{mn}(a) \qquad (7)$$

The wavelength in the ϕ direction is dictated by m since $Y_n^m \propto e^{im\phi}$. Thus m counts the number of full wavelengths in the circumferential direction around the sphere.

The wavelength is largest around the equator, and diminishes as we move towards the poles. This would seem to imply the acoustics short circuit is greater at the poles producing stronger evanescence. Also the redial decay, given by the ratio of Hankel functions (Eq.9), is independent of latitude. Thus the wave function in the polar angle must somehow compensate for this increased acoustics short circuit near the poles. Therefore the wavelength in the θ direction depends on m, in such a way that the spherical harmonic Y_n^m has the same radial decay for all possible values of m, and the acoustic short circuit is equalized so that it is independent of latitude.

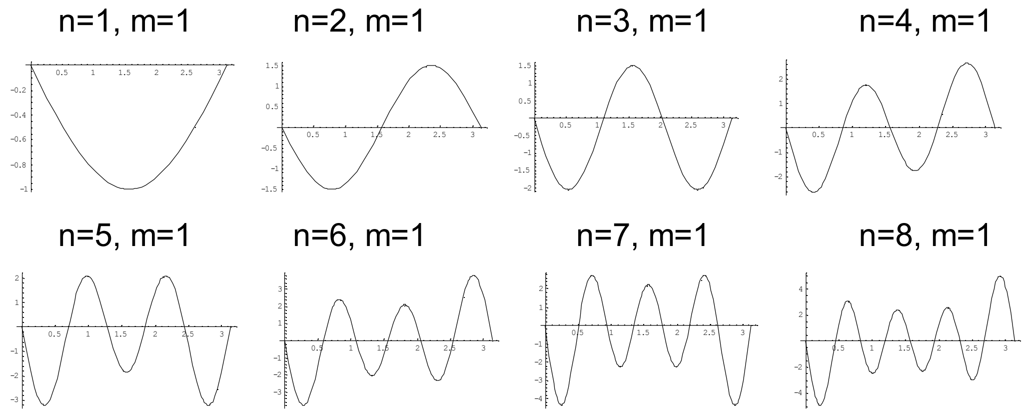


Fig. 7 Plots of the associated Legendre functions, $P_n^m(Cos\theta)$ as a function of θ for n=1, 2,3,4,5,6,7,8 and m=1

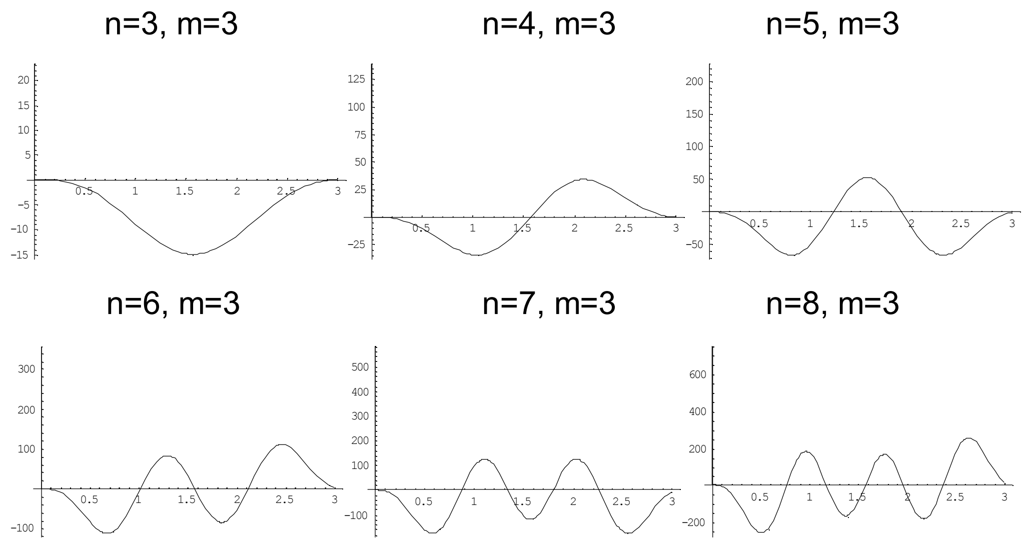


Fig.8 Plots of the associated Legendre functions $P_n^m(Cos\theta)$, as a function of θ for n=3, 4,5,6,7,8 and m=3

Figures 7 and 8 show that the dependency with respect to the polar angle θ . Here, as the pictures indicate, the number of “half-wavelengths” from $\theta = 0$ to $\theta = \pi$ is given by ($n - m + 1$).

These “half-wavelengths” are not strictly sinusoidal, as the figures show. At the poles the standing waves taper smoothly to zero. The half-wavelength near the poles is greater than the half-wavelengths at the equator. These two effects would appear to compensate to some degree for the increased acoustic short circuit due to the vanishing circumferential wavelength near the poles. Also note that for fixed n the number of these “half-wavelengths” increases when m decreases and vise versa.

Hankel functions dictate how the spherical wave spectrum components change phase and amplitude. The asymptotic behavior of evanescent waves by studying the behavior when $ka \ll n$ and $kr \ll n$.

$$\frac{h_n(kr)}{h_n(ka)} \approx (\frac{a}{r})^{n+1}, \qquad (8)$$

The power law decay similar to the cylindrical evanescent wave case. [Williams “Fourier Acoustics”pp129-132]

This evanescent behavior is greatest when small argument expansion for the Hankel functions is $r-a$. Since as r increases the condition $kr \ll n$ breaks down and we must take more terms in the small argument expansion for Hankel function. As r increases the decay of wave distribution goes slowly. The energy of azimuth waves transform to the energy of radial waves.

IX. PARAMETERS OF SOURCE

Consider case of vibrating spherical cap $\theta < \dfrac{\pi}{6}$ on the sphere (radius=1). Surface vibrates with velocity $U(\theta)e^{i\omega t}$, where U is any sort of function of θ. We first express the velocity amplitude $U(\theta)$ in terms of a series of Legendre functions.

["Theoretical Acoustic" P.M. Morse and K.U. Ingard pp.338-339]

$$U(\theta) = \sum_{m=0}^{\infty} U_m P_m(\cos\theta) \qquad (1)$$

$$U_m = (m + \frac{1}{2}) \int_0^{\pi} U(\theta) P_m(\cos\theta) \sin\theta d\theta \qquad (2)$$

The pressure wave expressed in a series

$$p = \sum_{m=0}^{\infty} A_m P_m(\cos\theta) h_m(kr) e^{-i\omega t} \qquad (3)$$

The values of the coefficients A_m must be determined in terms of the known coefficients U_m. The radial velocity of the air at the surface of the sphere is

$$u_a = \frac{1}{\rho c} \sum_{m=0}^{\infty} A_m B_m P_m(\cos\theta) e^{i\delta_m - i\omega t} \qquad (4)$$

where $\dfrac{d}{d\zeta} h_m(\zeta) = iB_m(\zeta) e^{i\delta_m(\zeta)} \qquad \zeta = ka = \dfrac{2\pi a}{\lambda} \qquad (5)$

$$B_m \simeq \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m-1)(m+1)}{(ka)^{m+2}} \qquad m > 0 \qquad (6)$$

$$\delta_m \simeq \frac{-m(ka)^{2m+1}}{1^2 \cdot 3^2 \cdot 5^2 \cdot \ldots \cdot (2m-1)^2 (2m+1)(m+1)} \qquad (7)$$

The radial velocity of the air at $r = a$ must equal that of the surface of the sphere, and equating coefficients of the two series, term by term, we obtain equations for the coefficients A_m in terms of U_m:

$$A_m = \frac{\rho c U_m}{B_m} e^{-i\delta_m} \qquad (8)$$

Pictures of analytical solution can be easily obtained, namely, acoustic pressure distribution (Eq.3) and velocity distribution (Eq.1).
Analytical solution is series from spherical functions also.

$$\sum_{n=0}^{\infty} C_n \psi_n(r) = \sum_{n=0}^{N} C_n \psi_n(r) + \sum_{n=N+1}^{\infty} C_n \psi_n(r)$$

Remainder $\sum_{n=N+1}^{\infty} C_n \psi_n(r)$ of the series should be very small.

Important thing, that how many terms sufficient to assume the convergence.
To achieve this goal it's necessary to analyse the remainder term. There are a lot of theoretical methods to analyse remainder term, for example Fourier, Taylor series expansions, however that methods are too complicated, because for solving acoustic inverse problem it is convenient to use eigenfunction method, namely, HELS method, which is basically experimental. Therefore it's better to analyse remainder numerically.

One way is to compare each term to first term and when the dependency becomes very small, for example, 10^{-5} or more, it can be cut down. And this finite series of analytical solution can be used for reconstruct acoustic field by using eigenfunction method (HELS).

X. NUMERICAL IMPLEMENTATION OF HELS METHOD

10.1 Vibrating spherical cap

The radiated acoustic pressures are obtained by means of an expansion of independent functions

$\psi_{m,n}(r,\theta,\varphi) = h_m(kr)P_{n,m}(\cos\theta)\begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}$ generated by the Gram-Schmidt orthnormalization with respect to the particular solutions to the Helmholtz equation on the vibrating surface.

The coefficients associated with these independent functions are determined by requiring the assumed form of the solution to satisfy the pressure boundary condition at the measurement points.

$$\rho c\sum_{n=1}^{N} C_i\psi^*_{mi}(x_m) = p^*_m(x_m)$$

$$\begin{bmatrix}\psi^*_{11} & \dots & \psi^*_{1N} \\ \dots & \dots & \dots \\ \psi^*_{M1} & \dots & \psi^*_{MN}\end{bmatrix}\begin{Bmatrix}C_1 \\ \dots \\ C_N\end{Bmatrix} \approx \begin{Bmatrix}p_{01} \\ \dots \\ p_{0N}\end{Bmatrix} = \begin{Bmatrix}p_1(x_m)+n(x_m) \\ \dots \\ p_1(x_m)+n(x_m)\end{Bmatrix}$$

$n(x_m)$ - Noise

The errors incurred in this process are minimized by the least-square-method.

$I = \sum_{m=1}^{M}[\rho c\sum_{n=1}^{N} C_n\psi^*_{mn} - p_{om}]^2$ Criterion function

$[\Gamma]\{C\} = \{B\}$ $\quad\quad$ $\Gamma_{mi} = \rho c\sum_{n=1}^{N}\psi^*_{nm}(x)\psi^*_{ni}(x)$ and $B_m = \sum_{n=1}^{N} p_{0n}(x)\psi^*_{nm}(x)$

$[\Gamma]^{\mu} = ([\psi^*_{mn}]^T[\psi^*_{mn}])^{-1}[\psi^*_{mn}]^T$ Pseudoinverse matrix

Once these coefficients are specified, the acoustic pressure at any points, including the source surface, is completely determined.

$$\{C\} = (\rho c)^{-1} [\Gamma\ ]^{\mu} \{p_0\}$$

In deriving equation for coefficients has no restriction have been imposed on the measurement points. They can be taken at any point in the field as long as they don't overlap each other.

The coefficients C_n are determined by requiring the assumed-form solution to satisfy the boundary condition $p_0(x_m, \omega)$ at the measurement points x_m

$$\rho c \sum_{n=0}^{N} C_n \psi_{mn}(x_m, \omega) = p_0(x_m, \omega)$$

$p(x, \omega) = \rho c \sum_{j=1}^{N} C_j \psi_j (x, \omega)$ - reconstructed acoustic pressure.

This investigation focuses on the optimal number of terms.

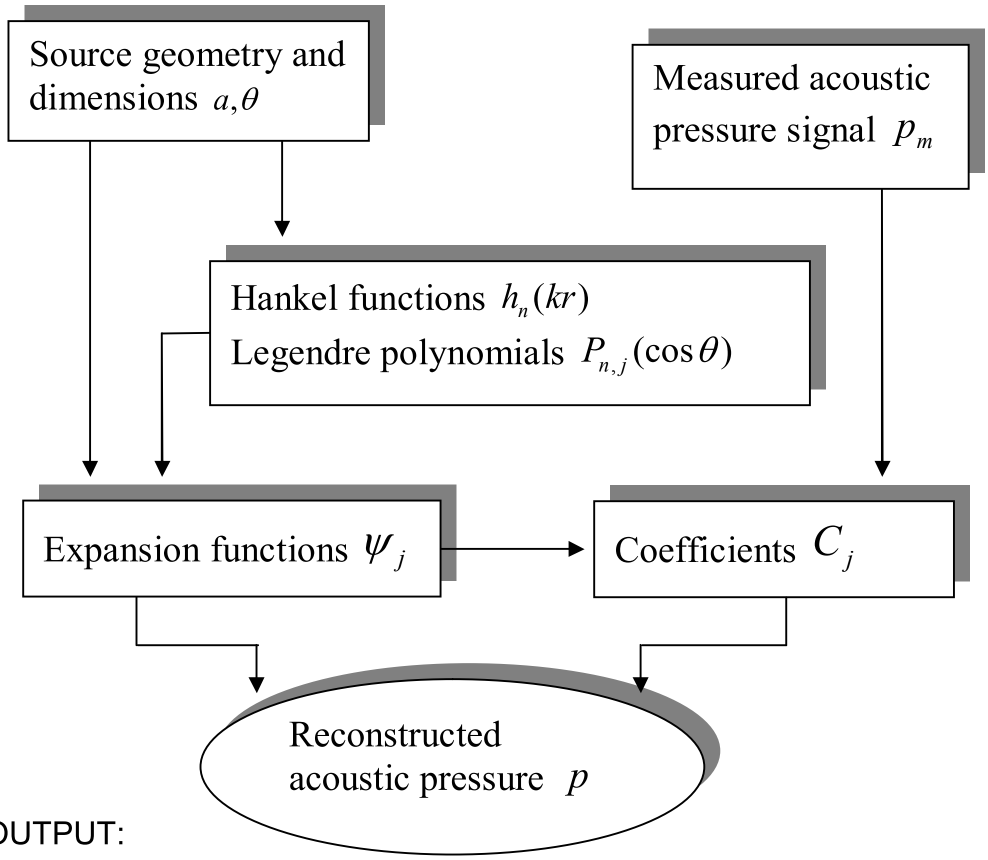
Function $I_0(N) = \min_{C_n} |p_0(r) - \psi_n(r) C_n|^2$ is minimizing of error.

Doctor Sean F. Su investigates error as function $I_1(N) = |p_{aux} - \psi(r_{aux})C|$. He uses the auxiliary measurement at r_{aux} and determines the optimal number of terms.

Alternative criterion may be $I_2(N) = \int_S |p(r) - \sum \psi_n(r) C_n|^2 dS$. The error $I_1(N)$ is discretized form of the error $I_2(N)$.

XI. ALGORITHM

INPUT:



OUTPUT:

$$p(x,\omega) = \rho c \sum_{j=1}^{N} C_j \psi_j(x,\omega)$$

Fig.9 Algorithm of reconstruction

XII.TEST SETUP

How many microphones and where it is necessary to take?
The spatial resolution of reconstruction depends on the frequency of interest. From signal analysis, we have learned that the sample rate must be greater than twice the highest frequency of interest in order to avoid aliasing. In practice, the sample rate is often selected to be 2.5 times the highest frequency of interest to make the process cost-effective. The wave number range is from 0.1 up to 20, therefore it is sufficient to take 50 microphones. Also the distance from the source surface is 0.2 or radius of source is 1m and distance from center to measurement points is 1.2. (See Fig.10) The measurement points can be taken at any point in the field as long as they don't overlap each other. Results have demonstrates that in the closer the measurement to a source, the more accurate the reconstructed acoustic pressure. Therefore we must try to take measurements in the near field.

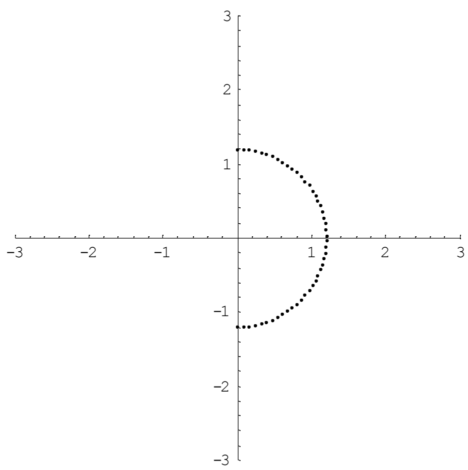


Fig. 10 Locations of microphones for axisymmetrical problem

Input data is analytical solution, which is infinite series, so it is necessary to check convergence of series.

To achieve this goal, we need to analyse the remainder term. If the remainder term is very small, then after certain value the series might be cut down.

One way is to compare each term to first term and when the dependency becomes very small, for example, 10^{-8} or more, it can be cut down. And this finite series of analytical solution can be used for reconstruct acoustic field by using eigenfunction method (HELS). Horizontal line (See Fig.11) shows, that remainder term is very small, less than 10^{-8}, so we can stop here and use this series of analytical solution.

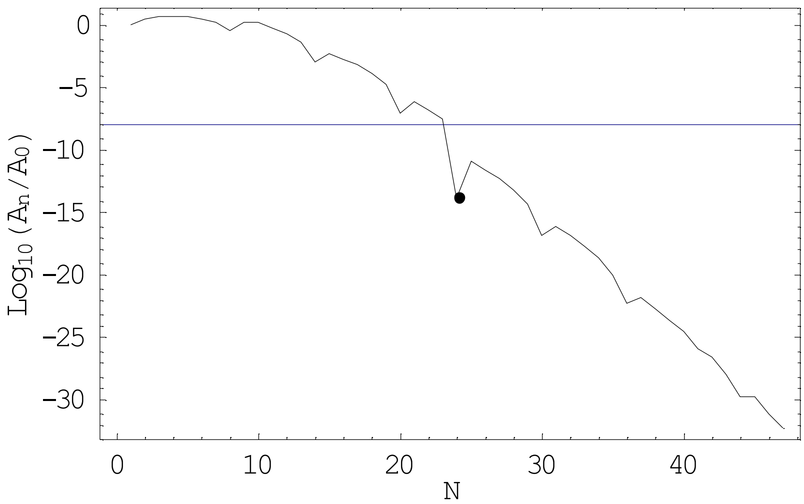


Fig.11 Convergence of analytical solution

Wave number, source geometry are parameters. After setting parameters we reconstruct acoustic pressure field.

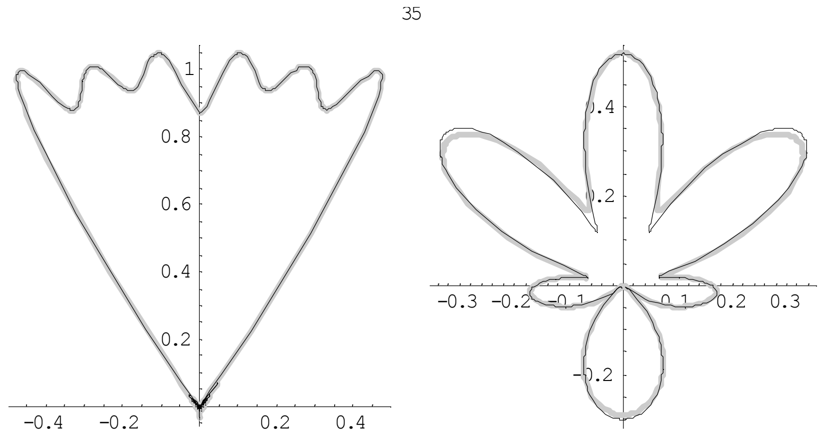


Fig.12 a) analytical velocity distribution; b) measurement pressure distribution

First of all obtain HELS coefficients (See Fig.13) and then reconstruct acoustic field by using spherical independent eigenfunctions.

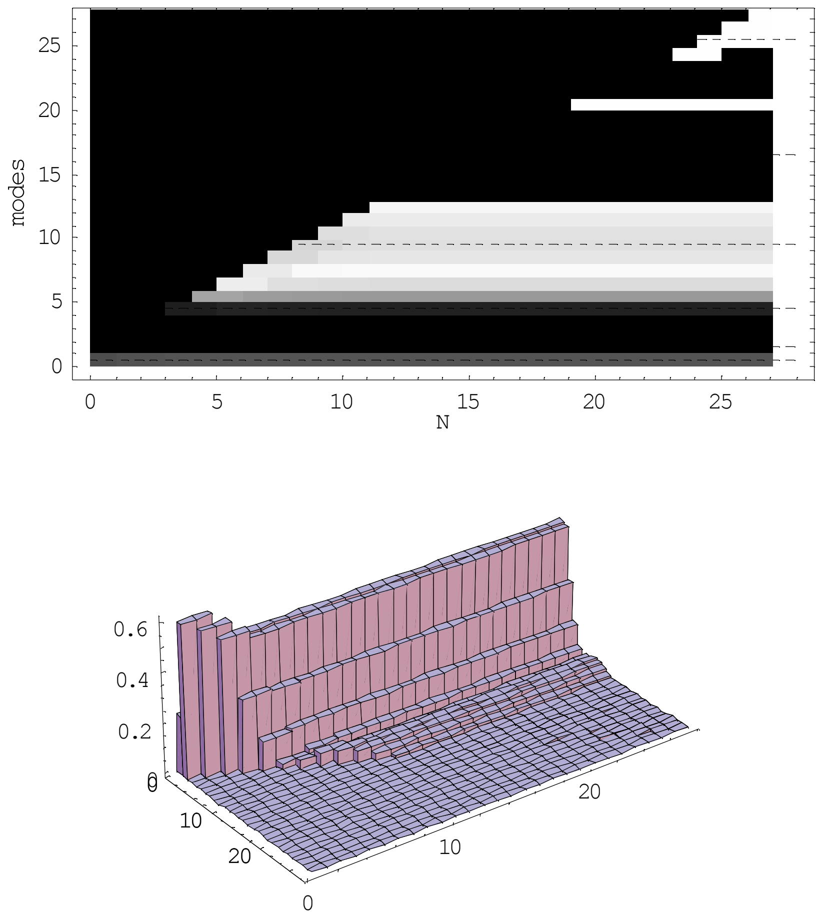


Fig.13 Spectrum of HELS coefficients

Once the coefficients C_n are solved, the surface acoustic pressure can be reconstructed. In fact, one can use Eq.3 to reconstruct acoustic pressures anywhere external to the vibrating surface.

It is emphasized that the HELS method if free of nonuniqueness difficulty, because it solves the Helmholtz equation directly. Hence the reconstructed surface acoustic pressures are always unique.

XIII. RESULTS AND DISCUSSION

13.1 Procedures

1). Take the data from the microphones. $p=\sum_{m=0}^{\infty} A_m P_m(\cos\theta) h_m(kr) e^{-i\omega t}$

$Uprec[\theta]=U_0(1-H(\theta-\theta_a))$ - velocity at any points on the radius of source.

$p_0^{'}$ -Input data

Noise model is $p_0^{'}=p_0[\theta]+n[\theta]=p_0+(NormalDistribution[\varepsilon, p_{rms}])$

$$SNR=10Lg(\frac{\int_0^{\pi}|p_0[\theta]|^2 d\theta}{\int_0^{\pi}|n[\theta]|^2 d\theta})=10\lg\frac{p_0[\theta]}{n[\theta]} \qquad \varepsilon=p_{rms}[\theta]\times 10^{-\frac{SNR}{10}}$$

$$p_{rms}[\theta]=\sqrt{\frac{1}{\pi}\int_0^{\pi}|p_{prec}[\theta]|^2 d\theta}$$

2). Reconstruct acoustic field. $p^{*}=\rho c\sum_{i=1}^{n} C_i \psi_i^{*}$

3). Obtain velocity distribution from the pressure.

$\mathrm{Re}\,cU_m=\sum_{m=0}^{N-1} C_N P_m(Cos\theta)$ - reconstructed velocity by using Legendre functions.

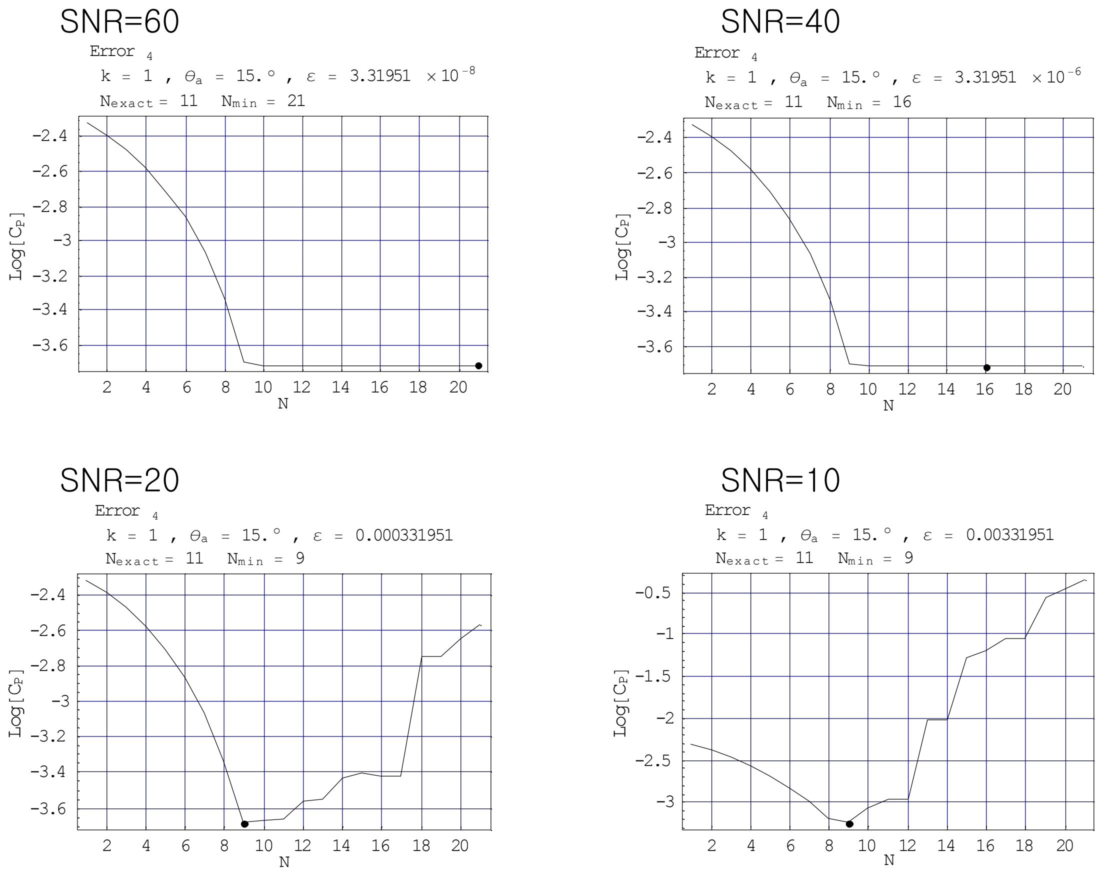
4). Determine the error function for velocity, analyze this function and made a conclusion for optimal number of expansion terms.

$$Error\_function = \int_{\partial S} \left|Uprec[\theta] - \mathrm{Re}\,cU[\theta]\right|^2 Sin[\theta]d\theta dr$$

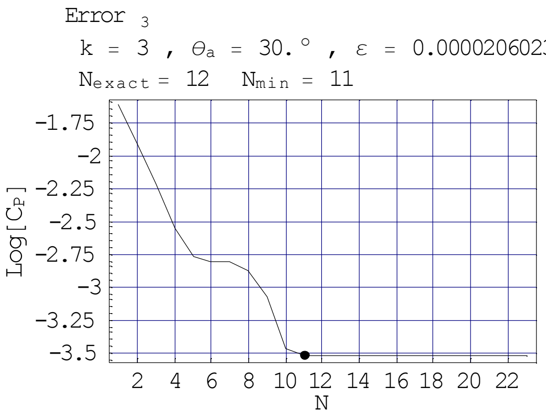
$$\theta \to \{0, \pi\}$$
$$r \to \{R_a, Rmic\}$$

Numerical results have demonstrated that the accuracy of reconstruction at SNR=60dB or 40dB, which means almost pure signal, on the measurement surface increases first with the number of expansion terms. However, the accuracy of reconstruction at SNR=10 or 20, which means noise level is high, increases first with the number of expansion terms up to certain value, then decreases monotonically thereafter. It is possible to set a systematic approach that enables one to select an optimal number of basis functions (See Fig.14) to reconstruct the radiated acoustic pressure. The presence of an optimal number of expansion terms seems to be reasonable. The optimal number depends on the measurement error and resolution.

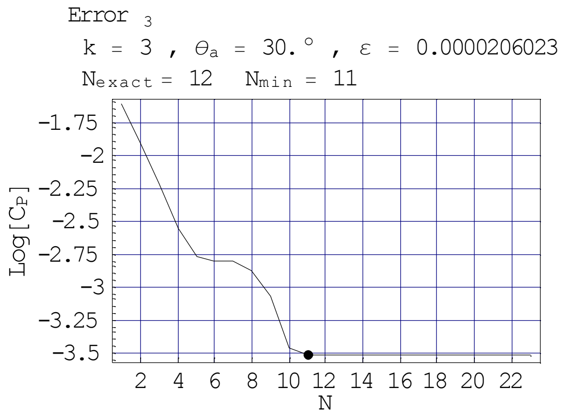
Fig.14 Alternative error functions in reconstruction versus the number of expansion terms



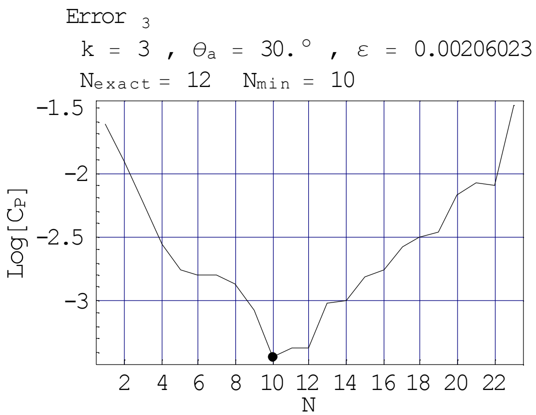
SNR=60



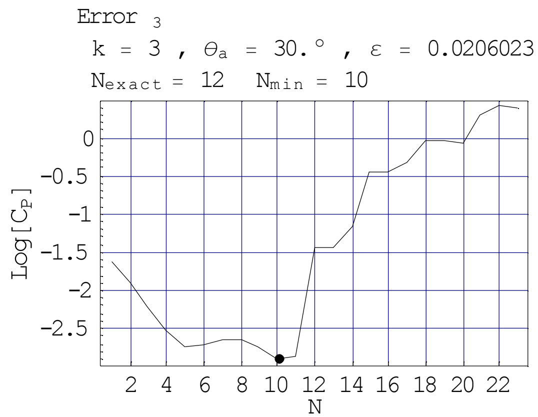
SNR =40



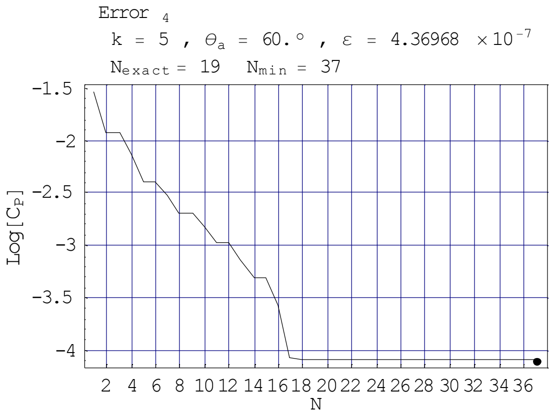
SNR=20



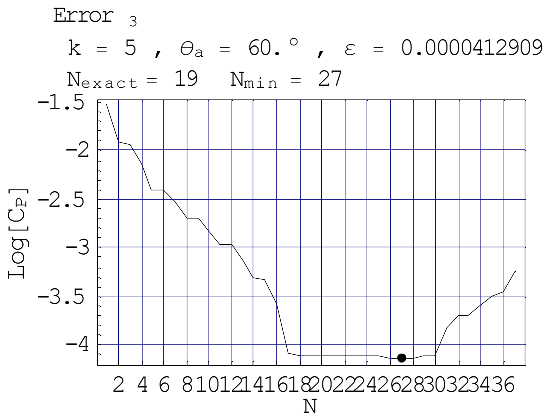
SNR=10



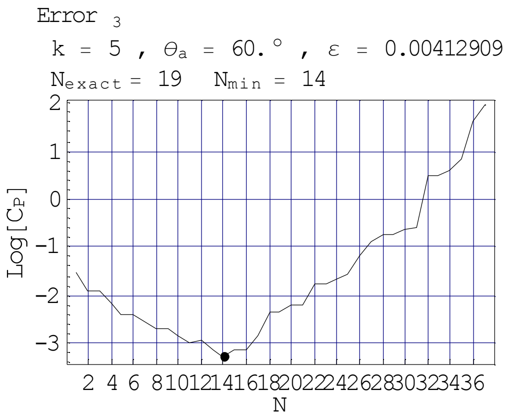
SNR=60



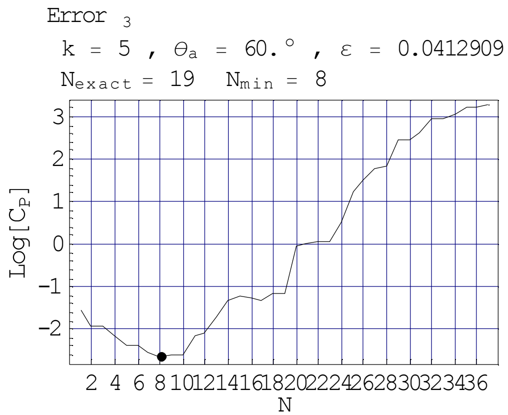
SNR=40



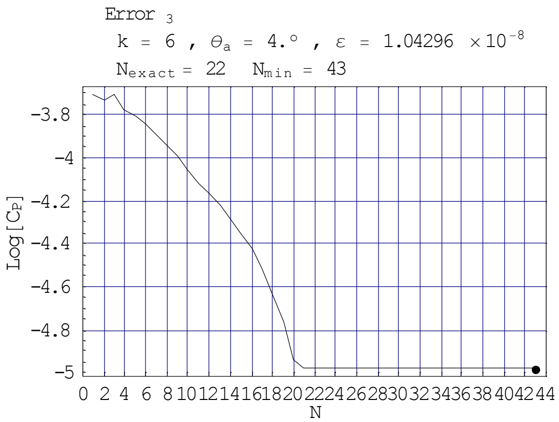
SNR=20dB



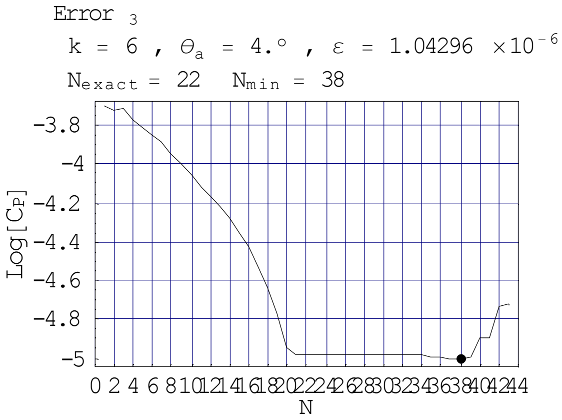
SNR=10dB



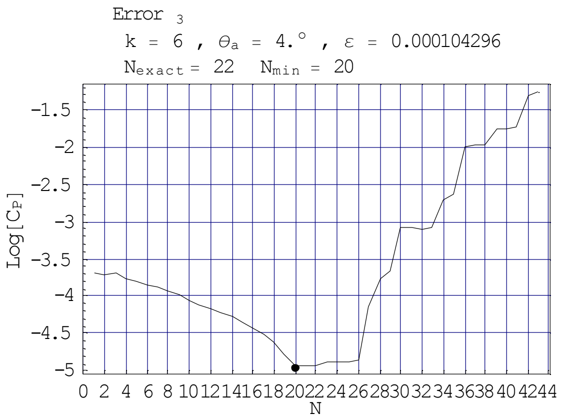
SNR=60



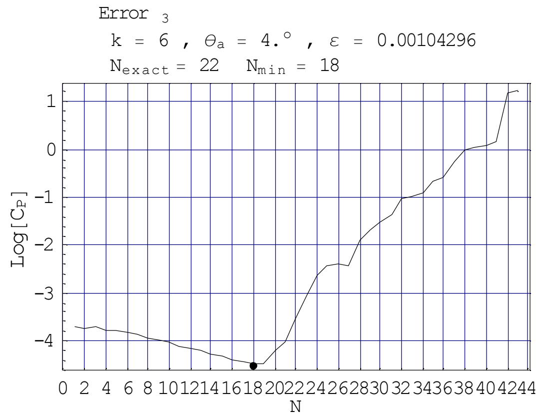
SNR=40



SNR=20



SNR=10

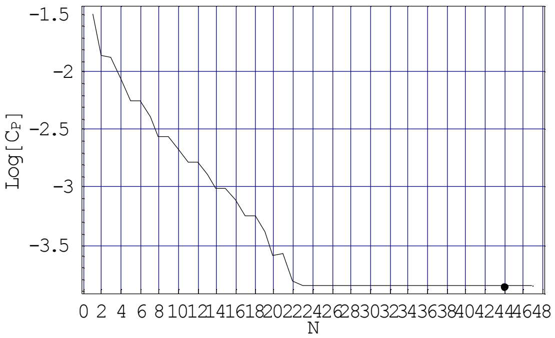


SNR=60

Error $_4$

$k = 8$, $\theta_a = 60.°$, $\varepsilon = 4.045 \times 10^{-7}$

$N_{exact} = 24 \quad N_{min} = 44$

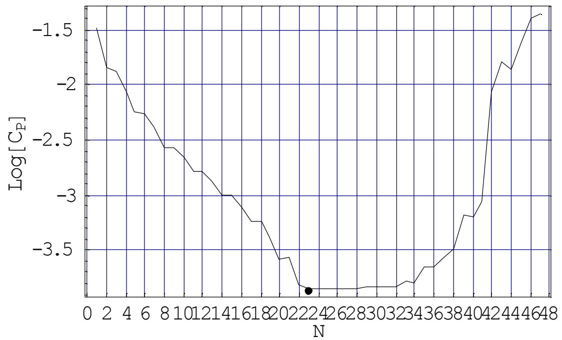


SNR=40

Error $_4$

$k = 8$, $\theta_a = 60.°$, $\varepsilon = 0.00004045$

$N_{exact} = 24 \quad N_{min} = 23$

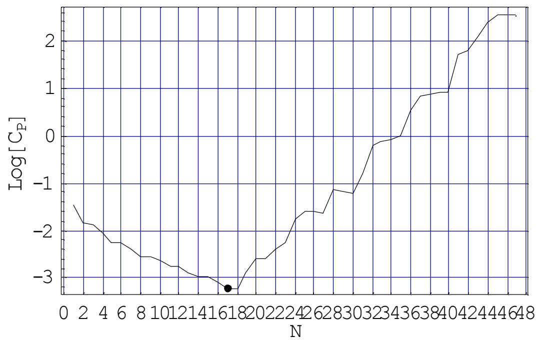


SNR=20

Error $_4$

$k = 8$, $\theta_a = 60.°$, $\varepsilon = 0.004045$

$N_{exact} = 24 \quad N_{min} = 17$

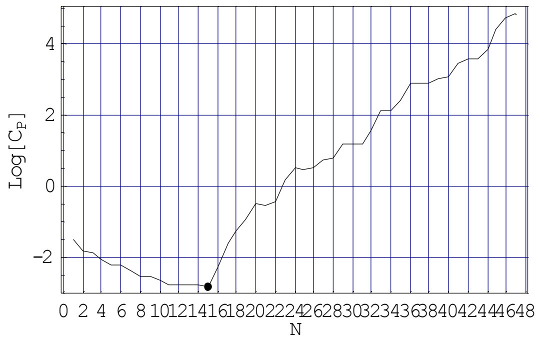


SNR=10

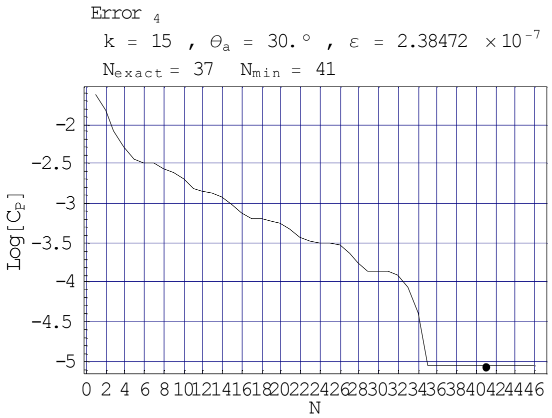
Error $_4$

$k = 8$, $\theta_a = 60.°$, $\varepsilon = 0.04045$

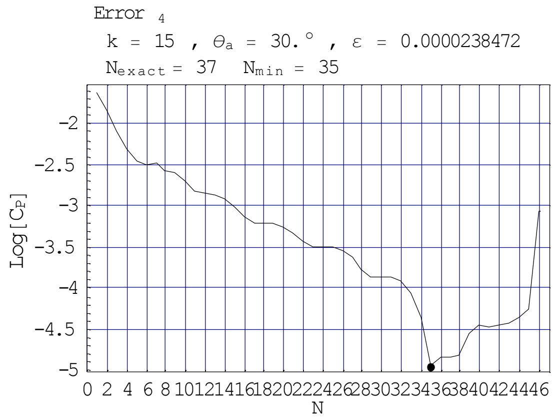
$N_{exact} = 24 \quad N_{min} = 15$



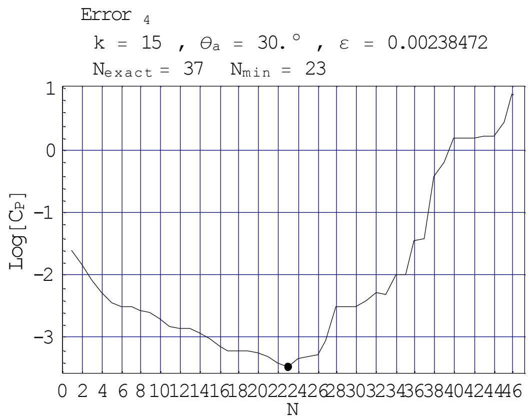
SNR=60



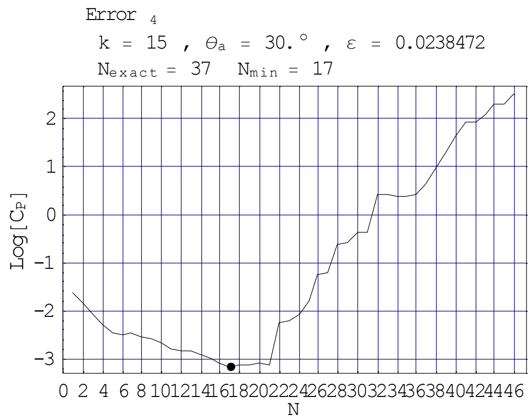
SNR=40

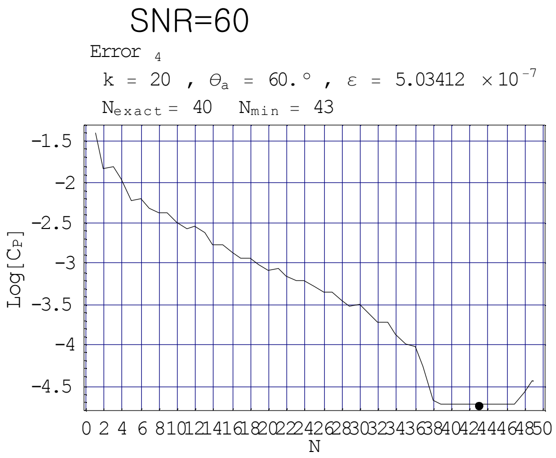


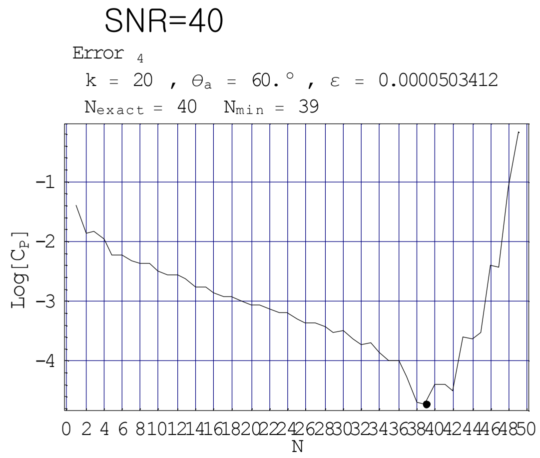
SNR=20

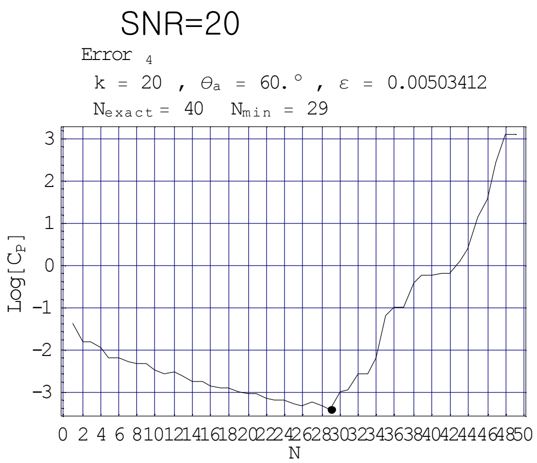


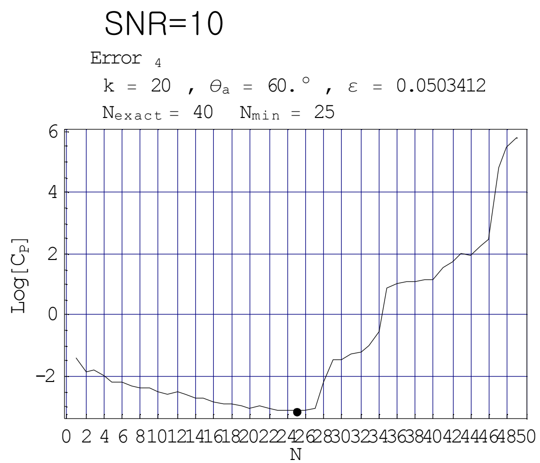
SNR=10




SNR=60
Error 4
k = 20 , θa = 60.° , ε = 5.03412 × 10^-7
Nexact = 40 Nmin = 43
Log[Cp]
N


SNR=40
Error 4
k = 20 , θa = 60.° , ε = 0.0000503412
Nexact = 40 Nmin = 39
Log[Cp]
N


SNR=20
Error 4
k = 20 , θa = 60.° , ε = 0.00503412
Nexact = 40 Nmin = 29
Log[Cp]
N


SNR=10
Error 4
k = 20 , θa = 60.° , ε = 0.0503412
Nexact = 40 Nmin = 25
Log[Cp]
N

13.2 Characteristic criterion:

1. The optimal number increases with the excitation frequency or wavenumber.

2. While $\lambda > R_a$, the optimal number, which depends on SNR, decreases very
slowly monotonically.

3. While $\lambda < R_a$, the wavenumber decreases monotonically.

4. From Nyquist theorem:

$$R_{mic} k_\theta \Delta\theta < \pi \;\; \Rightarrow \;\; Mp > \pi n$$

$$\frac{2\pi R_{mic}}{\pi} > n + \frac{1}{2}$$

13.3 Empirical criterion

$$N_{optimal}[k,\theta] = (0.901 + 5.07\sqrt{k} + 0.226k)(1 - 0.052\theta)$$

Figure 15 shows the behaviour of optimal number of terms, while angle of cap and wavenumber are fixed. The optimal number depends on wavenumber and doesn't depend on angle.

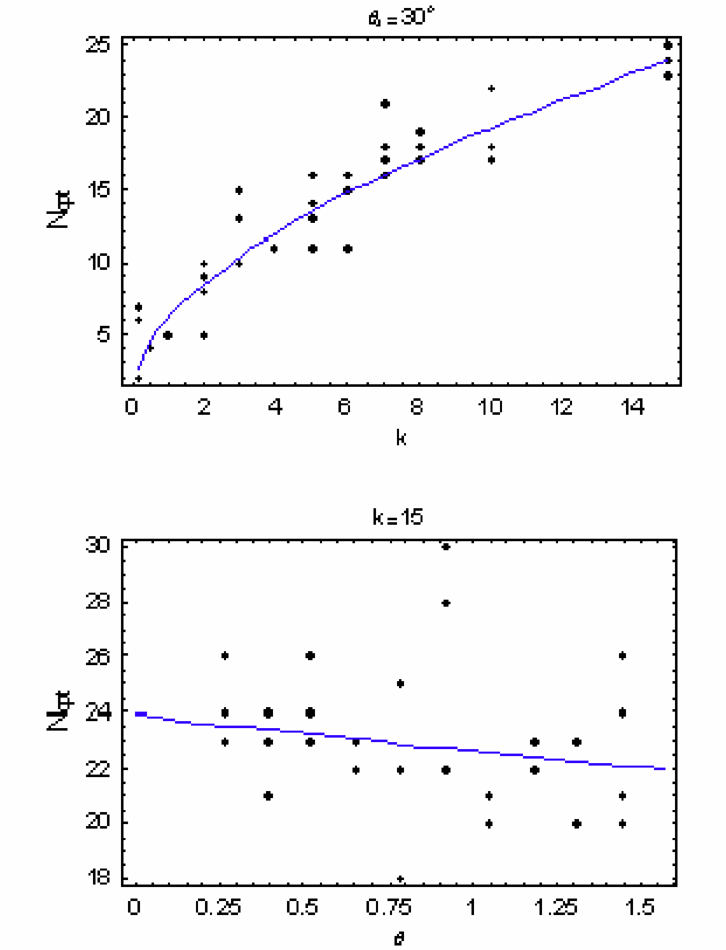


Fig.15 Empirical criterion

XIV. Criterion for the number of terms

What is relationship between wavenumber and optimal number of expansion terms?

1. The optimal number increases with the excitation frequency or wavenumber.

2. While $\lambda > R_a$, the optimal number, which depends on SNR, decreases very
slowly monotonically.

3. While $\lambda < R_a$, the optimal number decreases monotonically.

4. From Nyquist theorem

$$Rmic \cdot k_\theta \Delta\theta < \pi \quad \Rightarrow \quad Mp > \pi n$$

$$\frac{2k \cdot Rmic}{\pi} > n + \frac{1}{2}$$

5. The optimal number depends on wavenumber and source geometry.

References

[1] Wang, Z. (1995). "Helmholtz Equation-Least Squares (HELS) Method for Inverse acoustic problem"

[2] Zaoxi Wang and Sean F. Wu (1997) "Helmholtz equation-least-squares method for reconstructing the acoustic pressure field"

[3] Nassif Rayess and Sean Wu (2000) "Experimental validations of the HELS method for reconstructing acoustic radiation from a complex vibrating structure"

[4] Sean Wu (2000) "On reconstruction of acoustic pressure fields using the Helmholtz equation least square method"

[5] Morse, P.M., and Ingard, K.U (1968). "Theoretical Acoustics" (Mc Graw- Hill, New York), pp.332-366.

[6] Lawrence E. Kinsler (1980). "Fundamentals of acoustics"

[7] Miguel C. Junger and David Feit "Sound, structures and their interaction"

[8] Dowling and Ffowcs Williams "Sound and sources of sound"

[9] A. Korn, M. Korn "Mathematical handbook for scientists and engineers"

[10] Francis B. Hildebrand (1976) "Advanced calculus for applications"

[11] Donald E. Hall (1987) "Basic Acoustics" (California state University)

[12] D.G. Crighton, A. P. Dowling, J. E. Ffowcs Wiliams, M. Heckl and F.G. Leppington (1992) "Modern methods in analytical acoustics"

[13] Philip Denbigh (1998) "System Analysis & Signal Processing"

M.S. Thesis

Criterion for the number of terms in the eigenfunction expansion method for acoustic inverse problems

Nadezhda O. Kim

School of Mechanical Engineering

Graduate School

Yeungnam University

Directed by Professor Lim Byoung-Duk

Abstract

Eigenfunction expansion method, such as the HELS [Wang and Wu, 1997; Wu and Wang, 1998; Wu and Yu, 1998], is a mean of describing the acoustic field radiated from vibrating objects. The HELS method for acoustic holography uses spherical wave functions as a basis for the solution of the acoustic inverse problem.

The accuracy of reconstruction in expansion method increases up to certain number of terms, and then decreases thereafter.

Hence, there exist an optimal number of terms which gives the best reconstruction with minimum number of expansion functions. This optimum number depends on signal to noise ratio, frequency, stand-off distance, etc. Also the optimal number of terms is important to minimize the numerical computations involved in reconstruction, and make the reconstruction process most cost-effective. However, the optimal number has been determined by a heuristic method of searching the minimum-norm errors of the reconstructed acoustic pressures with respect to the measured data.

The purpose of this study is to develop a guideline to estimate an optimum number of expansion functions for a given set of measurements by using the notion of pseudo-wavenumber in spherical coordinates. The spherical pseudo-wavenumber is defined as a gradient of phase function with the introduction of the 2^{nd} kind Legendre functions. A systematic way of estimating the optimum number of expansion functions for the HELS method is developed.

Numerical examples are given for the acoustic field radiated from a sphere where a spherical cap is vibrating harmonically. The calculation results of optimal number of terms show good agreement with the predicted ones using the guidelines suggested in this study.

석사학위 논문

음향역문제 해석을 위한 고유함수전개법에 있어서 함수의 차수 최적화 방법

김 나디아
영남대학교 대학원
기계공학부

지도 교수: 임병덕

요약

HELS 와 같은 고유함수전개법은 진동물체로부터의 음향장을 기술하는 한 방법이다.
음향홀러그래피에 대한 HELS 법은 음향역문제 해석을 위한 기저함수로서 구면파함수를 사용한다. 함수의 차수를 증가시키면 고유함수전개법의 복원 정확도는 임의차수까지 증가되다가 그 이후에는 감소된다. 즉 복원 최소의 전개함수를 사용하여 가장 좋은

음원 복원을 얻을 수 있는 최적의 차수가 있다는 것이다. 이 최적차수는 신호대잡음비, 주파수, 음원으로부터 측정점까지의 거리 등에 의해 결정된다.

최적차수를 선정함은 음원복원에 수반되는 수치계산을 최소화시켜 복원과정을 가장 효율적으로 만들기 때문에 아주 중요하다. 하지만 지금까지 최적차수는 계측된 데이터에 대하여 복원된 음압의 최소 오차를 찾는 시행착요법이 사용되어왔다.

본 논문은 구면좌표계에서 의사 wavenumber 개념을 도입함으로써 어떤 주어진 계측자료에 대하여 확장함수의 최적차수를 예측하는 지침을 제시하고자 한다.

이 구면 의사-wavenumber 는 2 종 르장드르함수를 도입한 위상함수의 기울기로 정의되며 본 논문에서는 이것을 사용하여 HELS 법에서 전개함수의 최적차수를 예측하는 체계적인 방법을 제안하였다.

구면캡이 조화적으로 진동하는 구로부터 발생된 음향장에 대한 수치예를 보였으며.

수치예를 사용하여 계산한 최적차수는 본 논문의 최적차수 예측지침을 사용하여 예측한 것과 거의 일치하고 있다.