# Problem 1: Peskin-Schroeder, Problem 11-2

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## 1 Parts $a$ and $b$

The original lagrangian reads:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}[(\partial_\mu\phi^1)^2 + (\partial_\mu\phi^2)^2] + \frac{1}{2}\mu^2[(\phi^1)^2 + (\phi^2)^2] - \frac{\lambda}{4}[(\phi^1)^2 + (\phi^2)^2]^2 \\ &+\bar\psi(i\hat\partial)\psi - g\bar\psi(\phi^1 + i\gamma^5\phi^2)\psi\,. \end{aligned} \tag{1}$$

The global transformation

$$\begin{aligned} \phi^1 &\to \cos\alpha\ \phi^1 - \sin\alpha\ \phi^2\,, \\ \phi^2 &\to \sin\alpha\ \phi^1 + \cos\alpha\ \phi^2\,, \\ \psi &\to \exp(-i\alpha\gamma^5/2)\psi\,; \\ &\Longrightarrow\ \bar\psi \to \psi^+\exp(i\alpha\gamma^5/2)\gamma^0 = \bar\psi\exp(-i\alpha\gamma^5/2)\,. \end{aligned} \tag{2}$$

The first line of Eq.(1) is obviously invariant under $O(2)$ transformation (2). The second line is also invariant due to relations $\{\gamma^5,\gamma^\mu\} = 0\,;\quad \cos\alpha + i\gamma^5\sin\alpha = \exp(i\alpha\gamma^5)$.

After the symmetry is broken

$$\phi^i(x) = (v + \sigma(x),\ \pi) \tag{3}$$

the Lagrangian in terms of fields $\sigma$ and $\pi$ has the form:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 + \lambda v\sigma^3 + \lambda v\sigma^2\pi + \frac{\lambda}{4}(2\sigma^2\pi^2 + \sigma^4 + \pi^4) \\ &+\ \bar\psi[\hat p - gv]\psi - g\ \bar\psi\psi\sigma - g\ \bar\psi(i\gamma^5)\psi\pi\,. \end{aligned} \tag{4}$$

The mass of the fermion is apparently $m_0 = gv$.

## 2 Correction to the mass

Let's first see if the one-loop corrections to (4) are indeed finite. We first introduce new counterterms to the original lagrangian in the form:

$$\delta_Z\bar\psi\hat p\psi - \delta_g\bar\psi(\phi^1 + i\gamma^5\phi^2)\psi\,. \tag{5}$$

After symmetry breaking this leads to three types of counterterms:

$$\begin{aligned} i(\hat p\delta_Z - \delta_g v) &\quad \text{for fermion two-point function}\,; \\ -i\delta_g &\quad \text{for fermion-sigma vertex}\,; \\ -i\delta_g\cdot i\gamma^5 &\quad \text{for fermion-pion vertex}\,; \end{aligned} \tag{6}$$

There are two irredusible self-energy diagrams for fermion plus counterterm. To the $g^2$ order, the mass shift is given by

$$\delta m \equiv m - m_0 = \Sigma(p^2 = m^2) \approx \Sigma(p^2 = m_0^2)\,,$$

where $p$ is the fermion momentum and $m_0 = gv$ is tree level fermion mass. There are also tadpole diagrams but the requirement that $v$ is not shifted leads to $\langle\sigma\rangle = 0$, which assures the absence of tadpole contributions.

The calculation of the diagram with sigma particle gives:

$$\begin{aligned}
(-i)\Sigma_\sigma &= (-ig)^2 \int \frac{d^dk}{(2\pi)^d} \frac{(\hat{k} + m_0)}{[k^2 - m_0^2][(p-k)^2 - 2\mu^2]} \\
&= -g^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\hat{k} + m_0}{[k^2 - 2pkx - x2\mu^2 - (1-x)m_0^2 + p^2x]^2} \\
&= -g^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{\hat{\ell} + x\hat{p} + m_0}{[\ell^2 - \Delta]^2}\,,
\end{aligned} \tag{7}$$

where $\ell = k - xp$ and $\Delta = x2\mu^2 + (1-x)m_0^2 - p^2x(1-x)$. Evaluation at $\hat{p} = m_0$ gives:

$$\Sigma_\sigma = g^2 m_0 \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dx \frac{1 + x}{[m_0^2(1-x)^2 + 2\mu^2 x]^{2-\frac{d}{2}}}. \tag{8}$$

The diagram with pi-particle gives virtually the same. The numerator is now

$$i^2\gamma^5(\hat{k} + m_0)\gamma^5 = (\hat{k} - m_0)$$

and in $\Delta$ we now have infrared regular $\zeta^2$ instead of $2\mu^2$:

$$\Sigma_\pi = g^2 m_0 \frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}} \int_0^1 dx \frac{-1 + x}{[m_0^2(1-x)^2 + \zeta^2 x]^{2-\frac{d}{2}}}. \tag{9}$$

So, we see that expression $\Sigma_\sigma + \Sigma_\pi$ is *not* finite on its own. Thus the counterterm must contain an infinite part. Here comes the first problem: I don't see any other relation which would give me the value of $\delta_Z$. However, it seems to me that there exists a way around, which helps us at least to show the finiteness of fermion mass shift.

It comes from the complete one-loop analysis of the $\bar{\psi}\psi\sigma$ vertex. The one-loop corrections consist of 4 irreducible diagrams (2 with intermediate sigma and two with intermediate pi; in each pair the external sigma couples either to the fermion or scalar line). Then, we also have reducible diagrams: 2 self-energy diagrams for fermions plus self energy diagram for sigma. And of course counterterms. And probably one should take into account radiative diagrams to enforce infrared convergence.

Let's see where infinite contributions can come from. As we know, the sigma self energy has finite correction, even with fermion lines (tadpoles also have now fermion lines). So, the potentially infinity contributions can come from the fermion mass shift and from irreducible diagrams which would cancel (remember, that we want to keep $g$ fixed, so that the total one loop correction is not only finite, but even zero). However, below I show that irreducible diagrams have only finite contributions, therefore the mass shift must also be finite.

There are only two irreducible diagrams that produce potentially divergent terms: they are the diagrams with external sigma leg coupled to the fermion.

The diagram with intermediate sigma gives:

$$
\begin{aligned}
i\Gamma &= g^3 \int \frac{d^d k}{(2\pi)^d} \frac{(\hat{k}+m_0)(\hat{k}+m_0)}{[k^2-m_0^2]^2[(p-k)^2-2\mu^2]} \\
&= g^3 \int \frac{d^d k}{(2\pi)^d}(\hat{k}+m_0)(\hat{k}+m_0) \int_0^1 dx \int_0^{1-x} dy \frac{2}{[(k^2-m_0^2)(x+y)+((p-k)^2-2\mu^2)(1-x-y)]^3} \\
&= g^3 \int \frac{d^d k}{(2\pi)^d}(\hat{k}+m_0)(\hat{k}+m_0) \int_0^1 dz 2z \frac{1}{[k^2-zm_0^2-2\mu^2(1-z)+p^2z(1-z)-2pk(1-z))]^3}
\end{aligned}
\tag{10}
$$

Here $z = x+y$. After the shift of variable, the divergent contribution is given by $\ell^2$ term in the numerator. So, the divergent contribution (denoted by $\sim$) is

$$
\Gamma_\sigma \sim -ig^3 \int_0^1 dz 2z \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{[\ell^2-\Delta_\mu]^3} = g^3 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\frac{d}{2}\int_0^1 z dz \frac{1}{\Delta_\mu^{2-\frac{d}{2}}} \tag{11}
$$

Here $\Delta_\mu = m_0^2 z^2 + 2\mu^2(1-z)$. The same contribution from the intermediate pi diagram comes with opposite sign because in numerator we have

$$
i^2\gamma^5\hat{k}\hat{k}\gamma^5 = -k^2 .
$$

and also differs only by $\Delta_\zeta = m_0^2 z^2 + \zeta^2(1-z)$:

$$
\Gamma_\pi \sim g^3 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\frac{d}{2}\int_0^1 z dz \frac{-1}{\Delta_\zeta^{2-\frac{d}{2}}} . \tag{12}
$$

So, what we see is that their sum is finite:

$$
\begin{aligned}
\Gamma_\sigma + \Gamma_\pi &\sim g^3 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\frac{d}{2}\int_0^1 z dz \left[\frac{1}{\Delta_\mu^{2-\frac{d}{2}}} - \frac{1}{\Delta_\zeta^{2-\frac{d}{2}}}\right] \\
&= -g^3 \cdot 2 \cdot \frac{1}{(4\pi)^2}\int_0^1 z dz \log\left(\frac{\Delta_\mu}{\Delta_\zeta}\right) .
\end{aligned}
\tag{13}
$$

The expession is finite and, by the way, infrared stable.

However, I then understood that I still have the vertex counterterm. Therefore, I cannot say that finiteness of irreducible pieces leads to finiteness of the mass shift. It looks like I cannot avoid computing directly $\delta_Z$ and $\delta_g$. This is where I am stuck now.