Problem 1: Peskin-Schroeder, Problem 11-2

by Igor Ivanov **Version 2**

1 Parts a and b

The original lagrangian reads:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}[(\partial_\mu\phi^1)^2 + (\partial_\mu\phi^2)^2] + \frac{1}{2}\mu^2[(\phi^1)^2 + (\phi^2)^2] - \frac{\lambda}{4}[(\phi^1)^2 + (\phi^2)^2]^2 \\ &+\bar{\psi}(i\hat{\partial})\psi - g\bar{\psi}(\phi^1 + i\gamma^5\phi^2)\psi\,. \end{aligned} \tag{1}$$

The global transformation

$$\begin{aligned}\phi^1 &\to \cos\alpha\ \phi^1 - \sin\alpha\ \phi^2\,, \\ \phi^2 &\to \sin\alpha\ \phi^1 + \cos\alpha\ \phi^2\,, \\ \psi &\to \exp(-i\alpha\gamma^5/2)\psi\,; \\ &\Longrightarrow \bar{\psi} \to \psi^+\exp(i\alpha\gamma^5/2)\gamma^0 = \bar{\psi}\exp(-i\alpha\gamma^5/2)\,. \end{aligned} \tag{2}$$

The first line of Eq.(1) is obviously invariant under $O(2)$ transformation (2). The second line is also invariant due to relations $\{\gamma^5,\gamma^\mu\} = 0\,;\quad \cos\alpha + i\gamma^5\sin\alpha = \exp(i\alpha\gamma^5)$.

After the symmetry is broken

$$\phi^i(x) = (v + \sigma(x),\ \pi) \tag{3}$$

the Lagrangian in terms of fields σ and π has the form:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 + \lambda v\sigma^3 + \lambda v\sigma^2\pi + \frac{\lambda}{4}(2\sigma^2\pi^2 + \sigma^4 + \pi^4) \\ &+\ \bar{\psi}[\hat{p} - gv]\psi - g\ \bar{\psi}\psi\sigma - g\ \bar{\psi}(i\gamma^5)\psi\pi\,. \end{aligned} \tag{4}$$

The mass of the fermion is apparently $m_0 = gv$.

2 Correction to the mass

After writing down all counterterms (see, for instance, my version 1 or Matthew's solution), we end up with the following relevant counterterms (I adopt Matthew's notation):

$$\bar{\psi}(\delta_2\cdot\hat{p} - gv\delta_3) - g\delta_3\bar{\psi}(i\gamma^5)\psi \tag{5}$$

Thus the mass relation is now:

$$m - gv(1+\delta_3) = 0\,. \tag{6}$$

In my previous version I was inclined to think that the mass relation would be $m = gv\cdot\delta_3/\delta_2$ as if it were determined by setting (5) to zero. This supposition did not have any ground in fact. Then my second thought was that $m = gv \cdot (1+\delta_3)/(1+\delta_2)$. However, it again comes from just setting the total lagrangian to zero, which is not precisely the definition of the mass (as a pole of the two-point function). Second, we know that δ_2, which is responsible for wave function renormalization, cannot shift the mass. So, at this stage I am still puzzled. Matthew's solution does not convince me, though it looks like his formula (6) is correct. I will accept it for granted for now.

The value of δ_3 can be extracted from the requirement that $\bar\psi\psi\pi$ vertex receives no correction. There are two one loop irreducible diagrams that contribute to $\bar\psi\psi\pi$ vertex: when a fermion radiates off a sigma or pion then couples to the external pion leg and then takes back the radiated sigma or pion. Also we have the counterterm in the form $g\delta_3\gamma^5$. For our calculations, in both cases the external pion leg carries zero momentum.

The diagram with intermediate sigma gives:

$$\begin{aligned}\Gamma_\sigma &= \int \frac{d^dk}{(2\pi)^d}\frac{(-ig)\gamma^5 \cdot i(\hat k + m_0)\cdot g\gamma^5 \cdot i(\hat k+m_0)\cdot(-ig)\gamma^5\cdot i}{[k^2-m_0^2]^2[(p-k)^2-2\mu^2]} \\ &= -ig^3\gamma^5\int\frac{d^dk}{(2\pi)^d}\frac{(\hat k+m_0)(\hat k - m_0)}{[k^2-m_0^2]^2[(p-k)^2-2\mu^2]} \\ &= -ig^3\gamma^5\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-m_0^2][(p-k)^2-2\mu^2]}\,. \end{aligned} \tag{7}$$

It is interesting to see that the numerator has canceled one of the fermion propagators. This does not occur in the corrections to the $\bar\psi\psi\sigma$ vertex. Though technically it is clearly the consequence of the γ^5 factor, **it is not clear intuitively why this cancellation should take place**. Does pseudoscalar-fermion vertex has any special property or is it just a coincidence and a bare result of math? I wish I knew the answer.

OK, let's go further. Using Feynman parameters, one ca cast (7) in the form:

$$\begin{aligned}\Gamma_\sigma &= -ig^3\gamma^5\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-m_0^2x-2\mu^2(1-x)-2pk(1-x)+p^2(1-x)]^2} \\ &= g^3\gamma^5\int_0^1 dx\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\frac{1}{\Delta_\mu^{2-\frac{d}{2}}}\,, \end{aligned} \tag{8}$$

with

$$\Delta_\mu = m_0^2x + 2\mu^2(1-x) - p^2x(1-x) = m_0^2x^2 + 2\mu^2(1-x)\,.$$

The second diagram is done absolutely analogously. In this case we have only opposite sign due to i^2 and another Δ:

$$\Gamma_\pi = -g^3\gamma^5\int_0^1 dx\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\frac{1}{\Delta_\zeta^{2-\frac{d}{2}}}\,, \tag{9}$$

with

$$\Delta_\zeta = m_0^2x + \zeta^2(1-x) - p^2x(1-x) = m_0^2x^2 + \zeta^2(1-x)\,.$$

Here we introduced the infrared regulator: the vanishing pion mass ζ.

The sum of the two irreducible diagrams give the finite answer:

$$
\begin{aligned}
\Gamma_\sigma + \Gamma_\pi &= -g^3 \int_0^1 dx \frac{\Gamma(2-\frac{2}{2})}{(4\pi)^{d/2}} \left[\frac{1}{\Delta_\mu^{2-\frac{d}{2}}} - \frac{1}{\Delta_\zeta^{2-\frac{d}{2}}} \right] = -\frac{g^3}{16\pi^2} \int_0^1 \ln \frac{\Delta_\mu}{\Delta_\zeta} \\
&= -\frac{g^3}{16\pi^2} \int_0^1 \ln \frac{m_0^2 x^2 + 2\mu^2(1-x)}{m_0^2 x^2 + \zeta^2 (1-x)} \,. \qquad (10)
\end{aligned}
$$

We see that the result is infrared stable so we can put $\zeta \to 0$. Besides, it must be understood that the sigma mass squared $2\mu^2$ must be also taken at the tree level (it will receive corrections at one loop - this would be another example of a zeroth-order natural relation). So, the final answer for the mass shift reads:

$$
m = gv \left[1 + \frac{g^2}{16\pi^2} \int_0^1 dx \ln \left(1 + \frac{2\lambda}{g^2} \frac{1-x}{x^2} \right) \right] \,. \qquad (11)
$$