Review

- $\underline{x}^T A\underline{x} = \sum_{i=1}^{n}\sum_{j=1}^{n}[A]_{i,j} x_i x_j$

CHARCTERISTIC FUNCTION

- $\Phi_X(u) = E\, e^{iuX} = \int_{-\infty}^{\infty} e^{iuX} f_X(x)dx$ or $\sum_X p_x e^{iux}$
  - $|\Phi_X(u)| \le 1 = \Phi_X(0)$
  - $Y = aX+b \Rightarrow \Phi_Y(u) = e^{iub}\Phi_X(au)$
  - $f_X(x) = \frac{1}{2p}\int_{-\infty}^{\infty} e^{-iux}\Phi_X(u)du$

- $\Phi_{\underline{X}}(\underline{u}) = E\, e^{i\underline{u}^T\underline{X}} = E\, e^{i\sum_{k=1}^{n} u_k x_k} = \int e^{j\underline{u}^T\underline{x}} dF_{\underline{X}}(\underline{x}) = \int_{-\infty}^{\infty} dx_n \ldots \int_{-\infty}^{\infty} dx_1 e^{i\sum_{k=1}^{n} u_k x_k} f_{\underline{X}}(\underline{x})$
  - $|\Phi_{\underline{X}}(\underline{u})| \le 1 = \Phi_{\underline{X}}(\underline{0})$
  - $\underline{Y} = A\underline{X}+\underline{b} \Rightarrow \Phi_{\underline{Y}}(\underline{u}) = e^{i\underline{b}^T\underline{u}}\Phi_{\underline{X}}\left(A^T\underline{u}\right)$

$$\Phi_{\underline{Y}}(\underline{v}) = Ee^{j\underline{v}^T\underline{Y}} = Ee^{j\underline{v}^T(G\underline{X}+\underline{b})} = Ee^{j\underline{v}^T G\underline{X} + j\underline{v}^T\underline{b}} = e^{j\underline{v}^T\underline{b}}Ee^{j\underline{v}^T G\underline{X}}$$
$$= e^{j\underline{v}^T\underline{b}}Ee^{j\left(G^T\underline{v}\right)\underline{X}} = e^{j\underline{v}^T\underline{b}}\Phi_{\underline{X}}\left(G^T\underline{v}\right)$$

- $f_{\underline{X}}(\underline{x}) = \frac{1}{(2p)^n}\int e^{-j\underline{v}^T\underline{x}}\Phi_{\underline{X}}(\underline{v})d\underline{v}$
- $\Phi_{\underline{X}}(-\underline{u}) = \overline{\Phi_{\underline{X}}(\underline{u})}$

- $\Phi_{X_1,X_2}(u_1,u_2) = \Phi_{X_1,X_2,X_3}(u_1,u_2,0)$

- **Joint Moment** : $\frac{\partial^{\sum_{i=1}^{n} u_i}}{\partial v_1^{u_1}\partial v_2^{u_2}\cdots\partial v_n^{u_n}}\Phi_{\underline{X}}(\underline{0}) = j^{\sum_{i=1}^{n} u_i} E\left(\prod_{i=1}^{n} X_i^{u_i}\right)$

  Example

$$\Phi_i(\underline{0}) = jEX_i$$
$$\Phi_{ij}(\underline{0}) = j^2 E\left(X_i X_j\right)$$
$$\Phi_{ijk}(\underline{0}) = j^3 E\left(X_i X_j X_k\right)$$
$$\Phi_{ijk\ell}(\underline{0}) = j^4 E\left(X_i X_j X_k X_\ell\right)$$

$$\Phi_i(\underline{v}) = \frac{\partial}{\partial v_i}\Phi(\underline{v}) = \frac{\partial}{\partial v_i} Ee^{j(v_1X_1+\ldots+v_nX_n)} = E\frac{\partial}{\partial v_i}e^{j(v_1X_1+\ldots+v_nX_n)} = E\left(jX_ie^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$= jE\left(X_ie^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$\Phi_i(\underline{0}) = jE\left(X_ie^0\right) = jEX_i$$

$$\Phi_{ij}(\underline{v}) = \frac{\partial}{\partial v_j}\frac{\partial}{\partial v_i}\Phi(\underline{v}) = \frac{\partial}{\partial v_j}jE\left(X_ie^{j(v_1X_1+\ldots+v_nX_n)}\right) = jE\left(\frac{\partial}{\partial v_j}X_ie^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$= jE\left(X_i(jX_j)e^{j(v_1X_1+\ldots+v_nX_n)}\right) = j^2E\left(X_iX_je^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$\Phi_{ij}(\underline{0}) = j^2E\left(X_iX_je^0\right) = j^2E\left(X_iX_j\right)$$

$$\Phi_{ijk}(\underline{v}) = \frac{\partial}{\partial v_k}\frac{\partial}{\partial v_j}\frac{\partial}{\partial v_i}\Phi(\underline{v}) = \frac{\partial}{\partial v_k}j^2E\left(X_iX_je^{j(v_1X_1+\ldots+v_nX_n)}\right) = j^2E\frac{\partial}{\partial v_k}\left(X_iX_je^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$= j^2E\left(X_iX_j(jX_k)e^{j(v_1X_1+\ldots+v_nX_n)}\right) = j^3E\left(X_iX_jX_ke^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$\Phi_{ijk}(\underline{0}) = j^3E\left(X_iX_jX_ke^0\right) = j^3E\left(X_iX_jX_k\right)$$

$$\Phi_{ijk\ell}(\underline{v}) = \frac{\partial}{\partial v_\ell}\frac{\partial}{\partial v_k}\frac{\partial}{\partial v_j}\frac{\partial}{\partial v_i}\Phi(\underline{v}) = \frac{\partial}{\partial v_\ell}j^3E\left(X_iX_jX_ke^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$= j^3E\frac{\partial}{\partial v_\ell}\left(X_iX_jX_ke^{j(v_1X_1+\ldots+v_nX_n)}\right) = j^3E\left(X_iX_jX_k(jX_\ell)e^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$= j^4E\left(X_iX_jX_kX_\ell e^{j(v_1X_1+\ldots+v_nX_n)}\right)$$

$$\Phi_{ijk\ell}(\underline{0}) = j^4E\left(X_iX_jX_kX_\ell e^0\right) = j^4E\left(X_iX_jX_kX_\ell\right)$$

- **Central Moment** :

  Define $\Psi_{\underline{X}}(\underline{v}) = \ln\left(\Phi_{\underline{X}}(\underline{v})\right)$

  $$\Psi_i = \frac{\partial}{\partial v_i}\Psi_{\underline{X}}(\underline{0}),\ \Psi_{ij} = \frac{\partial^2}{\partial v_i \partial v_j}\Psi_{\underline{X}}(\underline{0})$$, and so on.

  - $Cov\left(X_i, X_j\right) = -\Psi_{ij}$
  - $E\left((X_i - EX_i)(X_j - EX_j)(X_k - EX_k)\right) = j^{-3}\Psi_{ijk}$
  - $E\left((X_i - EX_i)(X_j - EX_j)(X_k - EX_k)(X_\ell - EX_\ell)\right)$

    $= \Psi_{ijk\ell} + \Psi_{ij}\Psi_{k\ell} + \Psi_{ik}\Psi_{j\ell} + \Psi_{i\ell}\Psi_{jk}$

  Where we do not require that any or all of *i*, *j*, *k*, and ℓ be distinct.

Let

$$\Psi(\underline{v}) = \Psi_{\underline{X}}(\underline{v})$$

$$\Phi(\underline{v}) = \Phi_{\underline{X}}(\underline{v})$$

$$\Psi_i(\underline{v}) = \frac{\partial}{\partial v_i}\Psi(\underline{v}) \ , \ \Psi_{ij}(\underline{v}) = \frac{\partial^2}{\partial v_i \partial v_j}\Psi(\underline{v}) \text{ , and so on.}$$

$$\Psi_i = \frac{\partial}{\partial v_i}\Psi(\underline{0}) \ , \ \Psi_{ij} = \frac{\partial^2}{\partial v_i \partial v_j}\Psi(\underline{0}) \text{ , and so on.}$$

$$\Phi_i(\underline{v}) = \frac{\partial}{\partial v_i}\Phi(\underline{v}) \ , \ \Phi_{ij}(\underline{v}) = \frac{\partial^2}{\partial v_i \partial v_j}\Phi(\underline{v}) \text{ , and so on.}$$

First order:

$$\Psi_i(\underline{v}) = \frac{\partial}{\partial v_i}\Psi(\underline{v}) = \frac{\partial}{\partial v_i}\ln\left(\Phi(\underline{v})\right) = \frac{\frac{\partial}{\partial v_i}\Phi(\underline{v})}{\Phi(\underline{v})} = \frac{\Phi_i(\underline{v})}{\Phi(\underline{v})}$$

$$\Psi_i(\underline{0}) = \frac{\Phi_i(\underline{0})}{\Phi(\underline{0})} = \frac{jEX_i}{1} = jEX_i$$

Second order:

$$\Psi_{ij}(\underline{v}) = \frac{\partial}{dv_j}\frac{\Phi_i(\underline{v})}{\Phi(\underline{v})} = \frac{\Phi_{ij}(\underline{v})}{\Phi(\underline{v})} - \frac{\Phi_i(\underline{v})\Phi_j(\underline{v})}{\Phi^2(\underline{v})}$$

$$\Psi_{ij}(\underline{0}) = \frac{\Phi_{ij}(\underline{0})}{\Phi(\underline{0})} - \frac{\Phi_i(\underline{0})\Phi_j(\underline{0})}{\Phi^2(\underline{0})} = \frac{j^2E\left(X_iX_j\right)}{1} - \frac{\left(jEX_i\right)\left(jEX_j\right)}{1}$$

$$= j^2\left(E\left(X_iX_j\right) - EX_iEX_j\right) = -\left(E\left(X_iX_j\right) - EX_iEX_j\right)$$

$$\Psi_{ij}(\underline{0}) = -\left(E\left(X_iX_j\right) - EX_iEX_j\right) \qquad \ldots(1)$$

$$Cov\left(X_i, X_j\right) = E\left(\left(X_i - EX_i\right)\left(X_j - EX_j\right)\right)$$

$$= E\left(X_iX_j - X_i\left(EX_j\right) - \left(EX_i\right)X_j + EX_iEX_j\right)$$

$$= E\left(X_iX_j\right) - E\left(X_i\left(EX_j\right)\right) - E\left(\left(EX_i\right)X_j\right) + E\left(EX_iEX_j\right)$$

$$= E\left(X_iX_j\right) - EX_iEX_j - EX_iEX_j + EX_iEX_j$$

$$= E\left(X_iX_j\right) - EX_iEX_j$$

From (1),

$$Cov\left(X_i, X_j\right) = -\Psi_{ij}(\underline{0}) = -\Psi_{ij}$$

$$Cov\left(X_i, X_j\right) = -\Psi_{ij} \qquad \square$$

Third order:

$$\Psi_{ijk}(\underline{v}) = \frac{\partial}{dv_k}\left(\frac{\Phi_{ij}(\underline{v})}{\Phi(\underline{v})} - \frac{\Phi_i(\underline{v})\Phi_j(\underline{v})}{\Phi^2(\underline{v})}\right)$$

$$= \frac{\Phi_{ijk}(\underline{v})}{\Phi(\underline{v})} - \frac{\Phi_{ij}(\underline{v})\Phi_k(\underline{v})}{\Phi^2(\underline{v})} - \left(\frac{\Phi_{ik}(\underline{v})\Phi_j(\underline{v})}{\Phi^2(\underline{v})} + \frac{\Phi_i(\underline{v})\Phi_{jk}(\underline{v})}{\Phi^2(\underline{v})} - 2\frac{\Phi_i(\underline{v})\Phi_j(\underline{v})\Phi_k(\underline{v})}{\Phi^3(\underline{v})}\right)$$

$$= \frac{\Phi_{ijk}(\underline{v})}{\Phi(\underline{v})} - \frac{\Phi_{ij}(\underline{v})\Phi_k(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_{ik}(\underline{v})\Phi_j(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_i(\underline{v})\Phi_{jk}(\underline{v})}{\Phi^2(\underline{v})} + 2\frac{\Phi_i(\underline{v})\Phi_j(\underline{v})\Phi_k(\underline{v})}{\Phi^3(\underline{v})}$$

$$\Psi_{ijk}(\underline{0}) = \frac{\Phi_{ijk}(\underline{v})}{\Phi(\underline{v})} - \frac{\Phi_{ij}(\underline{v})\Phi_k(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_{ik}(\underline{v})\Phi_j(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_i(\underline{v})\Phi_{jk}(\underline{v})}{\Phi^2(\underline{v})} + 2\frac{\Phi_i(\underline{v})\Phi_j(\underline{v})\Phi_k(\underline{v})}{\Phi^3(\underline{v})}$$

$$= j^3E(X_iX_jX_k) - \left(j^2E(X_iX_j)\right)\left(jE(X_k)\right) - \left(j^2E(X_iX_k)\right)\left(jE(X_j)\right)$$

$$-\left(jE(X_i)\right)\left(j^2E(X_jX_k)\right) + 2\left(jE(X_i)\right)\left(jE(X_j)\right)\left(jE(X_k)\right)$$

$$\frac{\Psi_{ijk}(\underline{0})}{j^3} = E(X_iX_jX_k) - E(X_iX_j)E(X_k) - E(X_iX_k)E(X_j) \qquad \ldots(2)$$

$$-E(X_i)E(X_jX_k) + 2E(X_i)E(X_j)E(X_k)$$

$$E\left((X_i - EX_i)(X_j - EX_j)(X_k - EX_k)\right)$$

$$= E\left((X_i)(X_j)(X_k) - (X_i)(X_j)E(X_k) - (X_i)E(X_j)(X_k)\right.$$

$$+ (X_i)E(X_j)E(X_k) - E(X_i)(X_j)(X_k)$$

$$\left. + E(X_i)(X_j)E(X_k) + E(X_i)E(X_j)(X_k) - E(X_i)E(X_j)E(X_k)\right)$$

$$= E(X_iX_jX_k)$$

$$-E(X_iX_j)E(X_k) - E(X_iX_k)E(X_j) - E(X_i)E(X_jX_k)$$

$$+E(X_i)E(X_j)E(X_k) + E(X_i)E(X_j)E(X_k) + E(X_i)E(X_j)E(X_k)$$

$$-E(X_i)E(X_j)E(X_k)$$

$$= E(X_iX_jX_k) - E(X_iX_j)E(X_k) - E(X_iX_k)E(X_j) - E(X_i)E(X_jX_k)$$

$$+ 2E(X_i)E(X_j)E(X_k)$$

$$= \frac{\Psi_{ijk}(\underline{0})}{j^3} \qquad \text{from (2).}$$

$$= j^{-3}\Psi_{ijk}$$

$$E\left((X_i - EX_i)(X_j - EX_j)(X_k - EX_k)\right) = j^{-3}\Psi_{ijk} \qquad \square$$

Fourth order:

$$
\begin{aligned}
\Psi_{ijk\ell}(\underline{v}) &= \frac{\partial}{\partial \ell}\left(\begin{array}{l} \dfrac{\Phi_{ijk}(\underline{v})}{\Phi(\underline{v})} - \dfrac{\Phi_{ij}(\underline{v})\Phi_k(\underline{v})}{\Phi^2(\underline{v})} - \dfrac{\Phi_{ik}(\underline{v})\Phi_j(\underline{v})}{\Phi^2(\underline{v})} - \dfrac{\Phi_i(\underline{v})\Phi_{jk}(\underline{v})}{\Phi^2(\underline{v})} \\ +2\dfrac{\Phi_i(\underline{v})\Phi_j(\underline{v})\Phi_k(\underline{v})}{\Phi^3(\underline{v})} \end{array}\right) \\
&= \frac{\Phi_{ijk\ell}(\underline{v})}{\Phi(\underline{v})} - \frac{\Phi_{ijk}(\underline{v})\Phi_\ell(\underline{v})}{\Phi^2(\underline{v})} \\
&\quad -\left(\frac{\Phi_{ij\ell}(\underline{v})\Phi_k(\underline{v})}{\Phi^2(\underline{v})} + \frac{\Phi_{ij}(\underline{v})\Phi_{k\ell}(\underline{v})}{\Phi^2(\underline{v})} - 2\frac{\Phi_{ij}(\underline{v})\Phi_k(\underline{v})\Phi_\ell(\underline{v})}{\Phi^3(\underline{v})}\right) \\
&\quad -\left(\frac{\Phi_{ik\ell}(\underline{v})\Phi_j(\underline{v})}{\Phi^2(\underline{v})} + \frac{\Phi_{ik}(\underline{v})\Phi_{j\ell}(\underline{v})}{\Phi^2(\underline{v})} - 2\frac{\Phi_{ik}(\underline{v})\Phi_j(\underline{v})\Phi_\ell(\underline{v})}{\Phi^3(\underline{v})}\right) \\
&\quad -\left(\frac{\Phi_{i\ell}(\underline{v})\Phi_{jk}(\underline{v})}{\Phi^2(\underline{v})} + \frac{\Phi_i(\underline{v})\Phi_{jk\ell}(\underline{v})}{\Phi^2(\underline{v})} - 2\frac{\Phi_i(\underline{v})\Phi_{jk}(\underline{v})\Phi_\ell(\underline{v})}{\Phi^3(\underline{v})}\right) \\
&\quad +2\left(\begin{array}{l} \dfrac{\Phi_{i\ell}(\underline{v})\Phi_j(\underline{v})\Phi_k(\underline{v})}{\Phi^3(\underline{v})} + \dfrac{\Phi_i(\underline{v})\Phi_{j\ell}(\underline{v})\Phi_k(\underline{v})}{\Phi^3(\underline{v})} + \dfrac{\Phi_i(\underline{v})\Phi_j(\underline{v})\Phi_{k\ell}(\underline{v})}{\Phi^3(\underline{v})} \\ \qquad -3\dfrac{\Phi_i(\underline{v})\Phi_j(\underline{v})\Phi_k(\underline{v})\Phi_\ell(\underline{v})}{\Phi^4(\underline{v})} \end{array}\right) \\
&= \frac{\Phi_{ijk\ell}(\underline{v})}{\Phi(\underline{v})} \\
&\quad - \frac{\Phi_{ijk}(\underline{v})\Phi_\ell(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_{ij\ell}(\underline{v})\Phi_k(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_{ik\ell}(\underline{v})\Phi_j(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_i(\underline{v})\Phi_{jk\ell}(\underline{v})}{\Phi^2(\underline{v})} \\
&\quad - \frac{\Phi_{ij}(\underline{v})\Phi_{k\ell}(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_{ik}(\underline{v})\Phi_{j\ell}(\underline{v})}{\Phi^2(\underline{v})} - \frac{\Phi_{i\ell}(\underline{v})\Phi_{jk}(\underline{v})}{\Phi^2(\underline{v})} \\
&\quad + 2\frac{\Phi_{ij}(\underline{v})\Phi_k(\underline{v})\Phi_\ell(\underline{v})}{\Phi^3(\underline{v})} + 2\frac{\Phi_{ik}(\underline{v})\Phi_j(\underline{v})\Phi_\ell(\underline{v})}{\Phi^3(\underline{v})} + 2\frac{\Phi_i(\underline{v})\Phi_{jk}(\underline{v})\Phi_\ell(\underline{v})}{\Phi^3(\underline{v})} \\
&\qquad + 2\frac{\Phi_{i\ell}(\underline{v})\Phi_j(\underline{v})\Phi_k(\underline{v})}{\Phi^3(\underline{v})} + 2\frac{\Phi_i(\underline{v})\Phi_{j\ell}(\underline{v})\Phi_k(\underline{v})}{\Phi^3(\underline{v})} + 2\frac{\Phi_i(\underline{v})\Phi_j(\underline{v})\Phi_{k\ell}(\underline{v})}{\Phi^3(\underline{v})} \\
&\quad - 6\frac{\Phi_i(\underline{v})\Phi_j(\underline{v})\Phi_k(\underline{v})\Phi_\ell(\underline{v})}{\Phi^4(\underline{v})}
\end{aligned}
$$

$$\frac{\Psi_{ijk\ell}(\underline{0})}{j^4} = E\left(X_i X_j X_k X_\ell\right)$$
$$- E\left(X_i X_j X_k\right)E\left(X_\ell\right) - E\left(X_i X_j X_\ell\right)E\left(X_k\right)$$
$$- E\left(X_i X_k X_\ell\right)E\left(X_j\right) - E\left(X_i\right)E\left(X_j X_k X_\ell\right)$$
$$- E\left(X_i X_k\right)E\left(X_j X_\ell\right) - E\left(X_i X_\ell\right)E\left(X_j X_k\right) - E\left(X_i X_j\right)E\left(X_k X_\ell\right)$$
$$+ 2E\left(X_i X_j\right)E\left(X_k\right)E\left(X_\ell\right) + 2E\left(X_i X_k\right)E\left(X_j\right)E\left(X_\ell\right)$$
$$+ 2E\left(X_i\right)E\left(X_j X_k\right)E\left(X_\ell\right) + 2E\left(X_i X_\ell\right)E\left(X_j\right)E\left(X_k\right)$$
$$+ 2E\left(X_i\right)E\left(X_j X_\ell\right)E\left(X_k\right) + 2E\left(X_i\right)E\left(X_j\right)E\left(X_k X_\ell\right)$$
$$- 6E\left(X_i\right)E\left(X_j\right)E\left(X_k\right)E\left(X_\ell\right)$$

$$\Psi_{ijk\ell}(\underline{0}) = E\left(X_i X_j X_k X_\ell\right)$$
$$- E\left(X_i X_j X_k\right)E\left(X_\ell\right) - E\left(X_i X_j X_\ell\right)E\left(X_k\right)$$
$$- E\left(X_i X_k X_\ell\right)E\left(X_j\right) - E\left(X_i\right)E\left(X_j X_k X_\ell\right)$$
$$- E\left(X_i X_j\right)E\left(X_k X_\ell\right) - E\left(X_i X_k\right)E\left(X_j X_\ell\right) - E\left(X_i X_\ell\right)E\left(X_j X_k\right)$$
$$+ 2E\left(X_i X_j\right)E\left(X_k\right)E\left(X_\ell\right) + 2E\left(X_i X_k\right)E\left(X_j\right)E\left(X_\ell\right) + 2E\left(X_i\right)E\left(X_j X_k\right)E\left(X_\ell\right)$$
$$+ 2E\left(X_i X_\ell\right)E\left(X_j\right)E\left(X_k\right) + 2E\left(X_i\right)E\left(X_j X_\ell\right)E\left(X_k\right) + 2E\left(X_i\right)E\left(X_j\right)E\left(X_k X_\ell\right)$$
$$- 6E\left(X_i\right)E\left(X_j\right)E\left(X_k\right)E\left(X_\ell\right)$$

…(3)

$$\Psi_{ij}\Psi_{k\ell} + \Psi_{ik}\Psi_{j\ell} + \Psi_{i\ell}\Psi_{jk}$$
$$= (-1)\left(E\left(X_i X_j\right) - E\left(X_i\right)E\left(X_j\right)\right)(-1)\left(E\left(X_k X_\ell\right) - E\left(X_k\right)E\left(X_\ell\right)\right)$$
$$+ (-1)\left(E\left(X_i X_k\right) - E\left(X_i\right)E\left(X_k\right)\right)(-1)\left(E\left(X_j X_\ell\right) - E\left(X_j\right)E\left(X_\ell\right)\right)$$
$$+ (-1)\left(E\left(X_i X_\ell\right) - E\left(X_i\right)E\left(X_\ell\right)\right)(-1)\left(E\left(X_j X_k\right) - E\left(X_j\right)E\left(X_k\right)\right)$$
$$= \left(E\left(X_i X_j\right) - E\left(X_i\right)E\left(X_j\right)\right)\left(E\left(X_k X_\ell\right) - E\left(X_k\right)E\left(X_\ell\right)\right)$$
$$+ \left(E\left(X_i X_k\right) - E\left(X_i\right)E\left(X_k\right)\right)\left(E\left(X_j X_\ell\right) - E\left(X_j\right)E\left(X_\ell\right)\right)$$
$$+ \left(E\left(X_i X_\ell\right) - E\left(X_i\right)E\left(X_\ell\right)\right)\left(E\left(X_j X_k\right) - E\left(X_j\right)E\left(X_k\right)\right)$$
$$= E\left(X_i X_j\right)E\left(X_k X_\ell\right) + E\left(X_i X_k\right)E\left(X_j X_\ell\right) + E\left(X_i X_\ell\right)E\left(X_j X_k\right)$$
$$- E\left(X_i X_j\right)E\left(X_k\right)E\left(X_\ell\right) - E\left(X_i\right)E\left(X_j\right)E\left(X_k X_\ell\right) - E\left(X_j\right)E\left(X_\ell\right)E\left(X_i X_k\right)$$
$$- E\left(X_i\right)E\left(X_k\right)E\left(X_j X_\ell\right) - E\left(X_j\right)E\left(X_k\right)E\left(X_i X_\ell\right) - E\left(X_i\right)E\left(X_\ell\right)E\left(X_j X_k\right)$$
$$+ 3E\left(X_i\right)E\left(X_j\right)E\left(X_k\right)E\left(X_\ell\right)$$

…(4)

From (3) and (4),

$$
\begin{aligned}
&\boldsymbol{\Psi}_{ijk\ell} + \boldsymbol{\Psi}_{ij}\boldsymbol{\Psi}_{k\ell} + \boldsymbol{\Psi}_{ik}\boldsymbol{\Psi}_{j\ell} + \boldsymbol{\Psi}_{i\ell}\boldsymbol{\Psi}_{jk} \\
&\quad = E\left(X_i X_j X_k X_\ell\right) \\
&\qquad - E\left(X_i X_j X_k\right)E\left(X_\ell\right) - E\left(X_i X_j X_\ell\right)E\left(X_k\right) \\
&\qquad\quad - E\left(X_i X_k X_\ell\right)E\left(X_j\right) - E\left(X_i\right)E\left(X_j X_k X_\ell\right) \\
&\qquad + E\left(X_i X_j\right)E\left(X_k\right)E\left(X_\ell\right) + E\left(X_i X_k\right)E\left(X_j\right)E\left(X_\ell\right) + E\left(X_i\right)E\left(X_j X_k\right)E\left(X_\ell\right) \\
&\qquad\quad + E\left(X_i X_\ell\right)E\left(X_j\right)E\left(X_k\right) + E\left(X_i\right)E\left(X_j X_\ell\right)E\left(X_k\right) + E\left(X_i\right)E\left(X_j\right)E\left(X_k X_\ell\right) \\
&\qquad - 3E\left(X_i\right)E\left(X_j\right)E\left(X_k\right)E\left(X_\ell\right)
\end{aligned}
\qquad \ldots(5)
$$

$$
\begin{aligned}
&E\left(\left(X_i - EX_i\right)\left(X_j - EX_j\right)\left(X_k - EX_k\right)\left(X_\ell - EX_\ell\right)\right) \\
&\qquad = E\Big[X_i X_j X_k X_\ell \\
&\qquad\qquad - X_i X_j X_k E\left(X_\ell\right) - X_i X_j X_\ell E\left(X_k\right) - X_i X_k X_\ell E\left(X_j\right) - X_j X_k X_\ell E\left(X_i\right) \\
&\qquad\qquad + X_i X_j E\left(X_k\right)E\left(X_\ell\right) + X_i X_k E\left(X_j\right)E\left(X_\ell\right) + X_i X_\ell E\left(X_j\right)E\left(X_k\right) \\
&\qquad\qquad\quad + X_j X_k E\left(X_i\right)E\left(X_\ell\right) + X_j X_\ell E\left(X_i\right)E\left(X_k\right) + X_k X_\ell E\left(X_i\right)E\left(X_j\right) \\
&\qquad\qquad - X_i E\left(X_j\right)E\left(X_k\right)E\left(X_\ell\right) - X_j E\left(X_i\right)E\left(X_k\right)E\left(X_\ell\right) \\
&\qquad\qquad\quad - X_k E\left(X_i\right)E\left(X_j\right)E\left(X_\ell\right) - X_\ell E\left(X_i\right)E\left(X_j\right)E\left(X_k\right) \\
&\qquad\qquad + E\left(X_i\right)E\left(X_j\right)E\left(X_k\right)E\left(X_\ell\right)\Big] \\
&\qquad = E\left(X_i X_j X_k X_\ell\right) \\
&\qquad\quad - E\left(X_i X_j X_k\right)E\left(X_\ell\right) - E\left(X_i X_j X_\ell\right)E\left(X_k\right) \\
&\qquad\qquad - E\left(X_i X_k X_\ell\right)E\left(X_j\right) - E\left(X_j X_k X_\ell\right)E\left(X_i\right) \\
&\qquad\quad + E\left(X_i X_j\right)E\left(X_k\right)E\left(X_\ell\right) + E\left(X_i X_k\right)E\left(X_j\right)E\left(X_\ell\right) + E\left(X_i X_\ell\right)E\left(X_j\right)E\left(X_k\right) \\
&\qquad\qquad + E\left(X_j X_k\right)E\left(X_i\right)E\left(X_\ell\right) + E\left(X_j X_\ell\right)E\left(X_i\right)E\left(X_k\right) + E\left(X_k X_\ell\right)E\left(X_i\right)E\left(X_j\right) \\
&\qquad\quad - E\left(X_i\right)E\left(X_j\right)E\left(X_k\right)E\left(X_\ell\right) - E\left(X_j\right)E\left(X_i\right)E\left(X_k\right)E\left(X_\ell\right) \\
&\qquad\qquad - E\left(X_k\right)E\left(X_i\right)E\left(X_j\right)E\left(X_\ell\right) - E\left(X_\ell\right)E\left(X_i\right)E\left(X_j\right)E\left(X_k\right) \\
&\qquad\quad + E\left(X_i\right)E\left(X_j\right)E\left(X_k\right)E\left(X_\ell\right)
\end{aligned}
$$

$$
\begin{aligned}
&= E\left(X_i X_j X_k X_\ell\right) \\
&\quad - E\left(X_i X_j X_k\right) E\left(X_\ell\right) - E\left(X_i X_j X_\ell\right) E\left(X_k\right) \\
&\qquad - E\left(X_i X_k X_\ell\right) E\left(X_j\right) - E\left(X_j X_k X_\ell\right) E\left(X_i\right) \\
&\quad + E\left(X_i X_j\right) E\left(X_k\right) E\left(X_\ell\right) + E\left(X_i X_k\right) E\left(X_j\right) E\left(X_\ell\right) + E\left(X_i X_\ell\right) E\left(X_j\right) E\left(X_k\right) \\
&\qquad + E\left(X_j X_k\right) E\left(X_i\right) E\left(X_\ell\right) + E\left(X_j X_\ell\right) E\left(X_i\right) E\left(X_k\right) + E\left(X_k X_\ell\right) E\left(X_i\right) E\left(X_j\right) \\
&\quad - 3E\left(X_i\right) E\left(X_j\right) E\left(X_k\right) E\left(X_\ell\right)
\end{aligned}
$$

From (5),

$$
\begin{aligned}
&E\left(\left(X_i - EX_i\right)\left(X_j - EX_j\right)\left(X_k - EX_k\right)\left(X_\ell - EX_\ell\right)\right) \\
&\qquad = \Psi_{ijk\ell} + \Psi_{ij}\Psi_{k\ell} + \Psi_{ik}\Psi_{j\ell} + \Psi_{i\ell}\Psi_{jk}
\end{aligned}
$$

□

Joint Normality/Gaussian

Gaussian

- $f_X(x) = \frac{1}{\sqrt{2p}\, s} e^{-\frac{(x-m)^2}{2s^2}}$
- $\Phi_X(v) = e^{jvm_X - \frac{1}{2}v^2 s_X^2}$

$\underline{X}$ is jointly normal or **jointly Gaussian** with mean $\underline{m}$ and covariance matrix Λ if

- $\underline{X} \sim \mathcal{N}(\underline{m}_{\underline{X}}, \Lambda_{\underline{X}})$
- $\Leftrightarrow \Phi_{\underline{X}}(\underline{v}) = e^{j\underline{v}^T \underline{m} - \frac{1}{2}\underline{v}^T \Lambda \underline{v}}$
- $\Leftrightarrow f_{\underline{X}}(\underline{x}) = \frac{1}{(2p)^{\frac{n}{2}} \sqrt{\left|\det(\Lambda)\right|}} e^{-\frac{1}{2}(\underline{x}-\underline{m})^T \Lambda^{-1} (\underline{x}-\underline{m})}$

- $f_{X_{t_1}, X_{t_2}, \ldots X_{t_n}}(x_1, x_2, \ldots, x_n) = \frac{1}{(2p)^{\frac{n}{2}} |\Lambda|^{\frac{1}{2}}} e^{-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n} |\Lambda|_{ij}(x_i - m_i)(x_j - m_j)}$
- $f_{X_{t_1}, X_{t_2}}(x_1, x_2) = \frac{1}{2p s_{X_1} s_{X_2} \sqrt{1 - r^2}} e^{-\frac{\left(\frac{x_1 - EX_1}{s_{X_1}}\right)^2 - 2r\left(\frac{x_1 - EX_1}{s_{X_1}}\right)\left(\frac{x_2 - EX_2}{s_{X_2}}\right) + \left(\frac{x_2 - EX_2}{s_{X_2}}\right)^2}{2\left(1 - r^2\right)}}$,

  where $r = \frac{Cov(X_1, X_2)}{s_{X_1} s_{X_2}} = \frac{E(X_1 X_2) - EX_1 EX_2}{s_{X_1} s_{X_2}}$
- If $Cov(X_1, X_2) = 0$ (or $X_1 \perp\!\!\!\perp X_2$ which also make $Cov(X_1, X_2) = 0$),

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\times f_{X_2}(x_2) = \frac{1}{2p s_{X_1} s_{X_2}} e^{-\frac{\left(\frac{x_1-EX_1}{s_{X_1}}\right)^2+\left(\frac{x_2-EX_2}{s_{X_2}}\right)^2}{2}}$$

- Independent Gaussian random variables are always jointly Gaussian.

- Third or higher-order partial derivative of $\Psi(\underline{v})$ wrt. to components of $\underline{v}$ is zero.

  Ex. $\Psi_{ijk}(\underline{v}), \Psi_{ijk\ell}(\underline{v}) = 0$

$$\Psi(\underline{v}) = \ln(\Phi(\underline{v})) = \ln\left(e^{j\underline{v}^T\underline{m}-\frac{1}{2}\underline{v}^T\Lambda\underline{v}}\right) = j\underline{v}^T\underline{m} - \frac{1}{2}\underline{v}^T\Lambda\underline{v} = j\sum_a v_a m_a - \frac{1}{2}\sum_b\sum_a v_a v_b l_{ab}$$

The first term is linear, the second term is quadratic wrt. v.

So, any third or higher-order partial derivative of $\Psi(\underline{v})$ wrt. to components of $\underline{v}$ is zero.

- Third order joint Gaussian moments = 0

  $E\left[(X_i - EX_i)(X_j - EX_j)(X_k - EX_k)\right] = 0$

$E\left((X_i - EX_i)(X_j - EX_j)(X_k - EX_k)\right) = j^{-3}\Psi_{ijk} = 0$

- **Isserlis's Theorem**
  - Any forth-order central moment of jointly Gaussian r.v. is expressible as the sum of all possible products of pairs of their covariances.

    $E\left[(X_i - EX_i)(X_j - EX_j)(X_k - EX_k)(X_\ell - EX_\ell)\right]$

    $= Cov(X_i,X_j)Cov(X_k,X_\ell) + Cov(X_i,X_k)Cov(X_j,X_\ell) + Cov(X_i,X_\ell)Cov(X_j,X_k)$

  $E\left((X_i - EX_i)(X_j - EX_j)(X_k - EX_k)(X_\ell - EX_\ell)\right)$

  $= \cancel{\Psi_{ijk\ell}} + \Psi_{ij}\Psi_{k\ell} + \Psi_{ik}\Psi_{j\ell} + \Psi_{i\ell}\Psi_{jk}$

  and $Cov(X_i, X_j) = -\Psi_{ij}$

  - For zero mean jointly Gaussian vector,

    $E[X_1X_2X_3X_4] = E[X_1X_2]\times E[X_3X_4] + E[X_1X_3]\times E[X_2X_4] + E[X_1X_4]\times E[X_2X_3]$

    $E\left[(X_1 - EX_1)(X_2 - EX_2)(X_3 - EX_3)(X_4 - EX_4)\right] = E[X_1X_2X_3X_4]$

    $Cov(X_iX_j) = EX_iX_j - EX_iEX_j = EX_iX_j$

- Joint normality is preserved under **linear transformations**

  $\underline{Y} = G\underline{X} + \underline{b} \Rightarrow$

  - $\underline{Y} \sim \mathcal{N}(\underline{m}_{\underline{Y}}, \Lambda_{\underline{Y}})$
  - $\underline{m}_{\underline{Y}} = G\underline{m}_{\underline{X}} + \underline{b}$
  - $\Lambda_{\underline{Y}} = G\Lambda_{\underline{X}} G^T$

$$\Phi_{\underline{Y}}(\underline{v}) = e^{j\underline{v}^T\underline{b}}\Phi_{\underline{X}}(G^T\underline{v}) = e^{j\underline{v}^T\underline{b}}e^{j(G^T\underline{v})^T\underline{m}-\frac{1}{2}(G^T\underline{v})^T\Lambda(G^T\underline{v})}$$

$$= e^{j\underline{v}^T\underline{b}}e^{j\underline{v}^TG\underline{m}-\frac{1}{2}\underline{v}^TG\Lambda G^T\underline{v}} = e^{j\underline{v}^T(G\underline{m}+\underline{b})-\frac{1}{2}\underline{v}^T(G\Lambda G^T)\underline{v}}$$

- Every m^{th} order marginal of an n^{th} order jointly normal r.vec. is jointly normal

Proof :

choose $G_{m\times n}$ whose rows are unit vectors pointing in the directions of the appropriate coordinate axes.

$\underline{b} = \underline{0}$

- $\underline{X}$ is Jointly normal

⬄ every linear combination of its coordinates is a Gaussian r.v.

≡ For every real $\underline{v}$, $A = \underline{v}^T\underline{X}$ is normally distributed

Proof :

- 1) If $\underline{X} \sim \mathcal{N}(\underline{m}_{\underline{X}}, \Lambda_{\underline{X}})$ then $\underline{v}^T\underline{X}$ is normally distributed, $\forall\underline{v}$
  - $\underline{b} = \underline{0}$, $G = \underline{v}^T$
- 2) If $A = \underline{v}^T\underline{X}$ is normally distributed $\forall\underline{v}$ then $\underline{X} \sim \mathcal{N}(\underline{m}_{\underline{X}}, \Lambda_{\underline{X}})$
  - We have $A = \underline{v}^T\underline{X}$. So,

$$m_A = E\underline{v}^T\underline{X} = \underline{v}^T\underline{m}_{\underline{X}}$$

$$A - m_A = \underline{v}^T\underline{X} - \underline{v}^T\underline{m}_{\underline{X}} = \underline{X}^T\underline{v} - \underline{m}_{\underline{X}}^T\underline{v} = (\underline{X} - \underline{m}_{\underline{X}})^T\underline{v}$$

$$s_A^2 = E(A-m_A)^2 = E(A-m_A)(A-m_A) = E\left(\left((\underline{X}-\underline{m}_{\underline{X}})^T\underline{v}\right)^T\left((\underline{X}-\underline{m}_{\underline{X}})^T\underline{v}\right)\right)$$

$$= E\underline{v}^T(\underline{X}-\underline{m}_{\underline{X}})(\underline{X}-\underline{m}_{\underline{X}})^T\underline{v} = \underline{v}^T\left(E(\underline{X}-\underline{m}_{\underline{X}})(\underline{X}-\underline{m}_{\underline{X}})^T\right)\underline{v}$$

$$= \underline{v}^T\Lambda_{\underline{X}}\underline{v}$$

  - Because A is Gaussian, $\Phi_A(v) = e^{jvm_A-\frac{1}{2}v^2s_A^2}$ and

$$\Phi_A(1) = e^{jm_A-\frac{1}{2}s_A^2} = e^{j\underline{v}^T\underline{m}_{\underline{X}}-\frac{1}{2}\underline{v}^T\Lambda_{\underline{X}}\underline{v}} \quad (1)$$

  - By definition of characteristics function, $\Phi_A(v) = Ee^{jvA}$

$$\Phi_A(1) = Ee^{jA} = Ee^{j\underline{v}^T\underline{X}} \quad (2)$$

  - From (1) and (2), $Ee^{j\underline{v}^T\underline{X}} = e^{j\underline{v}^T\underline{m}_{\underline{X}}-\frac{1}{2}\underline{v}^T\Lambda_{\underline{X}}\underline{v}}$.

    Since this equation holds for every real $\underline{v}$, $Ee^{j\underline{v}^T\underline{X}} = \Phi_{\underline{X}}(\underline{v})$ and thus

    $\Phi_{\underline{X}}(\underline{v}) = e^{j\underline{v}^T\underline{m}_{\underline{X}}-\frac{1}{2}\underline{v}^T\Lambda_{\underline{X}}\underline{v}}$. So, $\underline{X}$ is jointly Gaussian.