Revision

- gaussian
  - $f_X(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$
  - $f_{X_{t_1}, X_{t_2}, \ldots X_{t_n}}(x_1, x_2, \ldots, x_n) = \dfrac{1}{(2\pi)^{\frac{n}{2}} |\Lambda|^{\frac{1}{2}}} e^{-\frac{1}{2|\Lambda|}\sum_{i=1}^{n}\sum_{j=1}^{n}|\Lambda|_{ij}(x_i - m_i)(x_j - m_j)}$
  - $f_{X_{t_1}, X_{t_2}}(x_1, x_2) = \dfrac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x_1 - EX_1}{\sigma_{X_1}}\right)^2 - 2\rho\left(\frac{x_1 - EX_1}{\sigma_{X_1}}\right)\left(\frac{x_2 - EX_2}{\sigma_{X_2}}\right) + \left(\frac{x_2 - EX_2}{\sigma_{X_2}}\right)^2}{2(1-\rho^2)}}$,

    where $\rho = \dfrac{Cov(X_1, X_2)}{\sigma_{X_1}\sigma_{X_2}} = \dfrac{E(X_1 X_2) - EX_1 EX_2}{\sigma_{X_1}\sigma_{X_2}}$
  - If $Cov(X_1, X_2) = 0$ (or $X_1 \perp\!\!\!\perp X_2$ which also make $Cov(X_1, X_2) = 0$),

    $$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) \times f_{X_2}(x_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}} e^{-\frac{\left(\frac{x_1 - EX_1}{\sigma_{X_1}}\right)^2 + \left(\frac{x_2 - EX_2}{\sigma_{X_2}}\right)^2}{2}}$$
  - Independent Gaussian random variables are always jointly Gaussian.

Complex Random Processes

- **Complex random variable** : Z = X + jY
  where X and Y are real random variable
- **Complex random process**: $\{Z(t)\} = \{X(t) + jY(t)\}$

- **Gaussian**
  - Z is a **complex Gaussian random variable** if X and Y are jointly Gaussian
  - {Z(t)} is a **complex Gaussian random process** if

    any vector $\underline{Z} = (Z(t_1), \ldots, Z(t_n)) = (X(t_1) + jY(t_1), \ldots, X(t_n) + jY(t_n))$

    is built from 2n real jointly Gaussian random variable.

- **Vanderkulk's Lemma** : The complex r.v. $Z = X + jY$ is zero mean and Gaussian, then

  $Ee^{jvZ} = e^{-\frac{1}{2}v^2 EZ^2}$

The complex r.v. $Z = X + jY$ is zero mean and Gaussian.

First, prove that X|Y ~ $N\left(\frac{s_X}{s_Y} r\, y, s_X^2\left(1-r^2\right)\right)$.

We have $f_{X,Y}(x,y)=\frac{1}{2p s_X s_Y\sqrt{1-r^2}}e^{-\frac{\left(\frac{x}{s_X}\right)^2-2r\left(\frac{xy}{s_X s_Y}\right)+\left(\frac{y}{s_Y}\right)^2}{2\left(1-r^2\right)}}$

where $r=\frac{Cov(X,Y)}{s_X s_Y}=\frac{E(XY)-EXEY}{s_X s_Y}=\frac{E(XY)}{s_X s_Y}$.

And $f_Y(y)=\frac{1}{\sqrt{2p}\,s_Y}e^{-\frac{y^2}{2s_Y^2}}$.

Then, $f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\frac{\frac{1}{2p s_X s_Y\sqrt{1-r^2}}e^{-\frac{\left(\frac{x}{s_X}\right)^2-2r\left(\frac{xy}{s_X s_Y}\right)+\left(\frac{y}{s_Y}\right)^2}{2\left(1-r^2\right)}}}{\frac{1}{\sqrt{2p}\,s_Y}e^{-\frac{y^2}{2s_Y^2}}}$

$$=\frac{1}{\sqrt{2p}\,s_X\sqrt{1-r^2}}e^{-\left(\frac{\left(\frac{x}{s_X}\right)^2-2r\left(\frac{xy}{s_X s_Y}\right)+\left(\frac{y}{s_Y}\right)^2}{2\left(1-r^2\right)}-\frac{y^2}{2s_Y^2}\right)}$$

Note that $\frac{\left(\frac{x}{s_X}\right)^2-2r\left(\frac{xy}{s_X s_Y}\right)+\left(\frac{y}{s_Y}\right)^2}{2\left(1-r^2\right)}-\frac{y^2}{2s_Y^2}$

$$=\frac{\left(\frac{x}{s_X}\right)^2-2r\left(\frac{xy}{s_X s_Y}\right)+\left(\frac{y}{s_Y}\right)^2-\left(1-r^2\right)\frac{y^2}{s_Y^2}}{2\left(1-r^2\right)}$$

$$=\frac{\left(\frac{x}{s_X}\right)^2-2r\left(\frac{xy}{s_X s_Y}\right)+r^2\frac{y^2}{s_Y^2}}{2\left(1-r^2\right)}=\frac{\left(\frac{x}{s_X}-\frac{r\,y}{s_Y}\right)^2}{2\left(1-r^2\right)}$$

Thus,

$$f_{X|Y}(x|y)=\frac{1}{\sqrt{2p}\,s_X\sqrt{1-r^2}}e^{-\left(\frac{\left(\frac{x}{s_X}-\frac{r\,y}{s_Y}\right)^2}{2\left(1-r^2\right)}\right)}=\frac{1}{\sqrt{2p}\,s_X\sqrt{1-r^2}}e^{-\left(\frac{\left(x-\frac{s_X}{s_Y}r\,y\right)^2}{2s_X^2\left(1-r^2\right)}\right)}$$

$= f_G(x)$;where G~$N\left(\frac{s_X}{s_Y} r\, y, s_X^2\left(1-r^2\right)\right)$

$$
\begin{aligned}
Ee^{jvZ} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{jv(x+jy)} f_{X,Y}(x,y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{jvx-vy} f_Y(y)\, f_{X|Y}(x|y)\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{jvx}e^{-vy} f_Y(y)\, f_{X|Y}(x|y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{jvx}e^{-vy} f_Y(y)\, f_G(x)\,dxdy \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{jvx} f_G(x)\,dx\right)e^{-vy} f_Y(y)\,dy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{jvg} f_G(g)\,dg\right)e^{-vy} f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty}\left(Ee^{jvG}\right)e^{-vy} f_Y(y)\,dy = \int_{-\infty}^{\infty} f_G(v)\,e^{-vy} f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty}\left(e^{jv\frac{\sigma_X}{\sigma_Y}\rho y}\, e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}\right)e^{-vy}\frac{1}{\sqrt{2\pi}\,\sigma_Y}e^{-\frac{y^2}{2\sigma_Y^2}}\,dy \\
&= e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}\frac{1}{\sqrt{2\pi}\,\sigma_Y}\int_{-\infty}^{\infty} e^{j\frac{\sigma_X}{\sigma_Y}\rho vy}e^{-vy}e^{-\frac{y^2}{2\sigma_Y^2}}\,dy = e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}\frac{1}{\sqrt{2\pi}\,\sigma_Y}\int_{-\infty}^{\infty} e^{j\frac{\sigma_X}{\sigma_Y}\rho vy}e^{-\left(\frac{y^2}{2\sigma_Y^2}+vy\right)}dy
\end{aligned}
$$

Consider the exponent,

$$
\frac{y^2}{2\sigma_Y^2}+vy = \frac{1}{2\sigma_Y^2}\left(y^2+2y\left(\sigma_Y^2 v\right)\right) = \frac{1}{2\sigma_Y^2}\left(\left(y+\sigma_Y^2 v\right)^2-\sigma_Y^4 v^2\right) = \frac{1}{2\sigma_Y^2}\left(y+\sigma_Y^2 v\right)^2-\frac{1}{2}v^2\sigma_Y^2
$$

Hence,

$$
\begin{aligned}
Ee^{jvZ} &= e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}\frac{1}{\sqrt{2\pi}\,\sigma_Y}e^{\frac{1}{2}v^2\sigma_Y^2}\int_{-\infty}^{\infty} e^{j\frac{\sigma_X}{\sigma_Y}\rho vy}e^{-\frac{1}{2\sigma_Y^2}\left(y+\sigma_Y^2 v\right)^2}dy \\
&= e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}e^{\frac{1}{2}v^2\sigma_Y^2}\int_{-\infty}^{\infty} e^{j\left(\frac{\sigma_X}{\sigma_Y}\rho v\right)y}\frac{1}{\sqrt{2\pi}\,\sigma_Y}e^{-\frac{1}{2\sigma_Y^2}\left(y+\sigma_Y^2 v\right)^2}dy \\
&= e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}e^{\frac{1}{2}v^2\sigma_Y^2}\int_{-\infty}^{\infty} e^{j\left(\frac{\sigma_X}{\sigma_Y}\rho v\right)y'}\frac{1}{\sqrt{2\pi}\,\sigma_Y}e^{-\frac{1}{2\sigma_Y^2}\left(y'+\sigma_Y^2 v\right)^2}dy' \\
&= e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}e^{\frac{1}{2}v^2\sigma_Y^2}\int_{-\infty}^{\infty} e^{j\left(\frac{\sigma_X}{\sigma_Y}\rho v\right)y'} f_{Y'}(y')\,dy'
\end{aligned}
$$

where $Y' \sim N\left(-\sigma_Y^2 v, \sigma_Y^2\right)$.

$$
\begin{aligned}
Ee^{jvZ} &= e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}e^{\frac{1}{2}v^2\sigma_Y^2} f_Y\left(\frac{\sigma_X}{\sigma_Y}\rho v\right) = e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}e^{\frac{1}{2}v^2\sigma_Y^2}e^{-j\left(\frac{\sigma_X}{\sigma_Y}\rho v\right)\left(-\sigma_Y^2 v\right)}e^{-\frac{1}{2}\sigma_Y^2\left(\frac{\sigma_X}{\sigma_Y}\rho v\right)^2} \\
&= e^{-\frac{1}{2}v^2\sigma_X^2(1-\rho^2)}e^{\frac{1}{2}v^2\sigma_Y^2}e^{-j\sigma_X\sigma_Y\rho v^2}e^{-\frac{1}{2}\sigma_X^2\rho^2 v^2} = e^{-\frac{1}{2}v^2\left(\sigma_X^2-\sigma_Y^2-2j\sigma_X\sigma_Y\rho\right)}
\end{aligned}
$$

$$
\begin{aligned}
EZ^2 &= E(X+jY)^2 = E\left(X^2-Y^2+j2XY\right) = EX^2-EY^2+2jEXY \\
&= \sigma_X^2-\sigma_Y^2-2j\sigma_X\sigma_Y\rho
\end{aligned}
$$

Therefore, $Ee^{jvZ} = e^{-\frac{1}{2}v^2 EZ^2}$ QED.

- Let W and Z be two complex random variable
  - $Cov(W,Z) = E(W-EW)\overline{(Z-EZ)} = \overline{Cov(Z,W)}$

- Z = X+jY
  - **mean** EZ = EX + jEY
  - **variance** $\sigma_Z^2 = Var(Z) = Cov(Z,Z) = E|Z-EZ|^2 = \sigma_X^2 + \sigma_Y^2$ (real)

$$\sigma_Z^2 = Var(Z) = Cov(Z,Z) = E(Z-EZ)\overline{(Z-EZ)} = E|Z-EZ|^2 \geq 0$$
$$= E\left[(X-EX)^2 + (Y-EY)^2\right] = VAR(X) + VAR(Y) = \sigma_X^2 + \sigma_Y^2$$

- Z = X+jY and z = x+jy

  X,Y iid. $\mathcal{N}(0,\sigma^2)$

  - $f_Z(z) = f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{|z|^2}{2\sigma^2}} = \frac{1}{\pi\sigma_Z^2}e^{-\frac{|z|^2}{\sigma_Z^2}}$

$$f_{X,Y}(x,y) = \left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}\right)\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}}\right) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Let z = x+jy

Then $x^2+y^2 = |z|^2$, so

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{|z|^2}{2\sigma^2}}$$

$f_Z(z)$ is a shorthand for $f_{X,Y}(x,y)$

$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2 = 2\sigma^2$

- Note: Though not as compact, any $f_{X,Y}(x,y)$ can be expressed only in terms of z by let

  $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2}$.

- $\{Z(t)\} = \{X(t) + jY(t)\}$
  - $m_Z(t) = m_X(t) + jm_Y(t)$
  - $K_Z(s,t) = Cov(Z(s),Z(t)) = E(Z(s)-m_Z(s))\overline{(Z(t)-m_Z(t))} = \overline{K_Z(t,s)}$ complex

$$K_Z(t,s) = E(Z(t)-m_Z(t))\overline{(Z(s)-m_Z(s))} = \overline{E\overline{(Z(t)-m_Z(t))}(Z(s)-m_Z(s))}$$
$$= \overline{E(Z(s)-m_Z(s))\overline{(Z(t)-m_Z(t))}} = \overline{K_Z(s,t)}$$

- If $K_Z(t,t-\tau)$ = function of τ only, we write $K_Z(t,t-\tau) = K_Z(\tau)$

- $R_Z(s,t) = EZ(s)\overline{Z(t)} = \overline{R_Z(s,t)}$
  - If $R_Z(t,t-t)$ = function of τ only, we write $R_Z(t,t-t) = R_Z(t)$
- If $m_Z(t) =$ constant and $R_Z(t,t-t) = R_Z(t)$, {Z(t)} is w.s.s.

Proper / circular / phase-invariant complex Gaussian

- $\underline{Z}$ is **proper (circular, phase-invariant) Gaussian** iff

  pseudo-covariance $E(Z_i - EZ_i)(Z_j - EZ_j) = 0$ for all 1 ≤ i,j ≤ n including when i = j

- same as $Cov(X_i, X_j) = Cov(Y_i, Y_j)$ and $Cov(X_i, Y_j) = -Cov(Y_i, X_j)$

$$E(Z_i - EZ_i)(Z_j - EZ_j) = E\left((X_i - EX_i) + j(Y_i - EY_i)\right)\left((X_j - EX_j) + j(Y_j - EY_j)\right)$$

$$= E\begin{pmatrix} (X_i - EX_i)(X_j - EX_j) - (Y_i - EY_i)(Y_j - EY_j) \\ + j(Y_i - EY_i)_i (X_j - EX_j) + j(X_i - EX_i)(Y_j - EY_j) \end{pmatrix}$$

$$= \left(E(X_i - EX_i)(X_j - EX_j) - E(Y_i - EY_i)(Y_j - EY_j)\right)$$

$$+ j\left(E(Y_i - EY_i)_i (X_j - EX_j) + E(X_i - EX_i)(Y_j - EY_j)\right)$$

$$= \left(Cov(X_i, X_j) - Cov(Y_i, Y_j)\right) + j\left(Cov(X_i, Y_j) + Cov(Y_i, X_j)\right)$$

$$= 0 + j0$$

- For E$\underline{Z}$ = $\underline{0}$ (E$\underline{X}$ = E$\underline{Y}$ = $\underline{0}$)

  $$EZ_iZ_j = E(X_i + jY_i)(X_j + jY_j) = E(X_iX_j - Y_iY_j + jY_iX_j + jX_iY_j)$$

  $$= (EX_iX_j - EY_iY_j) + j(EY_iX_j + EX_iY_j) = 0 + j0$$

  Then, $EX_iX_j - EY_iY_j = 0$ and $EY_iX_j + EX_iY_j = 0$

- $VAR(X_i) = VAR(Y_i)$

  $Cov(X_i, X_j) = Cov(Y_i, Y_j) \rightarrow Cov(X_i, X_i) = Cov(Y_i, Y_i)$

- $s_{Z_i}^2 = Var(Z_i) = E|Z_i|^2 = 2s_{X_i}^2 = 2s_{Y_i}^2$

  $s_{Z_i}^2 = Var(Z_i) = E|Z_i|^2 = s_{X_i}^2 + s_{Y_i}^2 = 2s_{X_i}^2 = 2s_{Y_i}^2$

- $Cov(X_i, Y_i) = 0$

  $Cov(X_i, Y_j) = -Cov(X_j, Y_i) \rightarrow Cov(X_i, Y_i) = -Cov(X_i, Y_i)$

- {Z(t) = X(t) + jY(t)} is **proper (circular, phase-invariant) Gaussian** iff

  all n-dimensional vector of the form $(Z(t_1), \ldots, Z(t_n))$ must be proper Gaussian

- Condition for properness: $E\left(Z(s)-m_Z(s)\right)\left(Z(t)-m_Z(t)\right) = 0$

- **n = 1**:Suppose
  - Z = X+jY is Gaussian
  - $s_X^2 = s_Y^2$
  - $Cov(X,Y) = 0$ (or X ⫫ Y which also make $Cov(X,Y) = 0$)

  Then, $f_{X,Y}(x,y) = \frac{1}{p s_Z^2} e^{-\frac{|z-m|^2}{s_Z^2}} = f_Z(z)$

$s_X^2 = s_Y^2 = \sigma^2 \Rightarrow s_Z^2 = s_X^2 + s_Y^2 = 2s^2$

$$f_{X,Y}(x,y) = \left(\frac{1}{\sqrt{2p}\,s} e^{-\frac{(x-EX)^2}{2s^2}}\right)\left(\frac{1}{\sqrt{2p}\,s} e^{-\frac{(y-EY)^2}{2s^2}}\right) = \frac{1}{2ps^2} e^{-\frac{(x-EX)^2+(y-EY)^2}{2s^2}}$$

$$= \frac{1}{p s_Z^2} e^{-\frac{(x-EX)^2+(y-EY)^2}{s_Z^2}}$$

Consider the exponent

$$(x-EX)^2+(y-EY)^2 = \left(x^2-2xEX+(EX)^2\right)+\left(y^2-2yEY+(EY)^2\right)$$

Let

- $m = m_Z = EX + jEY \Rightarrow |m_Z|^2 = (EX)^2+(EY)^2$
- z = x+jy ⇒ $|z|^2 = x^2+y^2$

Since $m\bar{z} + z\bar{m} = m\bar{z} + \overline{m\bar{z}} = 2\mathrm{Re}\left\{m\bar{z}\right\} = 2\mathrm{Re}\left\{(EX+jEY)(x-jy)\right\} = 2(xEX+yEY)$

$$(x-EX)^2+(y-EY)^2 = \left(x^2+y^2\right)+\left((EX)^2+(EY)^2\right)-\left(2xEX+2yEY\right)$$
$$= |z|^2+|m|^2-m\bar{z}-z\bar{m}$$

Since $|z-m|^2 = (z-m)\overline{(z-m)} = |z|^2+|m|^2-m\bar{z}-z\bar{m}$

$$(x-EX)^2+(y-EY)^2 = |z-m|^2$$

Hence, $f_{X,Y}(x,y) = \frac{1}{p s_Z^2} e^{-\frac{|z-m|^2}{s_Z^2}} = f_Z(z)$

- Let

  $\underline{Z} = (Z_1,\ldots,Z_n)^T = (X_1+jY_1,\ldots,X_n+jY_n)^T$

  $z_k = x_k + jy_k$ for $1 \le k \le n$ and $\underline{z} = (z_1,\ldots,z_n)^T$

  If

1) $\left(X_1,\ldots,X_n,Y_1,\ldots,Y_n\right)$ is *2n*-dimentional jointly Gaussian

2) $\underline{Z}$ is proper. $\Rightarrow$ $E\left(Z_j - EZ_j\right)\left(Z_k - EZ_k\right) = 0$ for all 1 ≤ j,k ≤ n including when j = k

then

$$f_{\underline{X},\underline{Y}}\left(\underline{x},\underline{y}\right) = f_{\underline{Z}}\left(\underline{z}\right) = \frac{1}{p^n \left|\det\left(K_Z\right)\right|} e^{-\left(\underline{z}-\underline{m}\right)^{\dagger} K_Z^{-1}\left(\underline{z}-\underline{m}\right)}$$

where

$\underline{m} = E\underline{Z}$ and

$K_Z = \left(Cov\left(Z_i, Z_j\right)\right)_{n\times n}$

- To see that it reduces correctly when n = 1,

$$f_Z\left(z\right) = \frac{1}{p^1 s_Z^2} e^{-\left(z-m\right)^{\dagger}\frac{1}{s_Z^2}\left(z-m\right)} = \frac{1}{p^1 s_Z^2} e^{-\frac{1}{s_Z^2}\overline{\left(z-m\right)}\left(z-m\right)} = \frac{1}{p^1 s_Z^2} e^{-\frac{1}{s_Z^2}\left|z-m\right|^2}$$

Only have to consider one case where j = k = 1

Obviously, X,Y is jointly Gaussian

So, have to show that

$E\left(Z_1 - EZ_1\right)\left(Z_1 - EZ_1\right) = E\left(Z_1 - EZ_1\right)^2 = E\left(Z - EZ\right)^2 = 0$ is equivalent to

$s_X^2 = s_Y^2$ and $Cov\left(X,Y\right) = 0$

This can be shown from

$$\begin{aligned} E\left(Z-EZ\right)^2 &= E\left(\left(X-EX\right) - j\left(Y-EY\right)\right)^2 \\ &= E\left(\left(\left(X-EX\right)^2 - \left(Y-EY\right)^2\right) - 2j\left(X-EX\right)\left(Y-EY\right)\right) \\ &= \left(s_X^2 - s_Y^2\right) - 2j\left(Cov\left(X,Y\right)\right) = 0 + j0 \end{aligned}$$

- For **n = 2**,

(Z1,Z2)^T is proper Gaussian iff

0) X_1, X_2, Y_1, Y_2 are jointly Gaussian
1) Z_1 is proper Gaussian
2) Z_2 is proper Gaussian
3) E$(Z_1 - EZ_1)(Z_2 - EZ_2)$ = 0

$$f_{X_1,Y_1,X_2,Y_2}\left(x_1,y_1,x_2,y_2\right) = \frac{1}{p^2 \det\left(\Lambda_Z\right)} e^{-\left(\underline{z}\right)^{\dagger}\Lambda_Z^{-1}\left(\underline{z}\right)}$$

where

$E\underline{Z} = 0$

$$r = \frac{1}{s_1 s_2} EZ_1\overline{Z_2}$$

$$\boxed{\begin{aligned}
&\Lambda_Z = E\underline{Z}\,\underline{Z}^{\dagger} = \begin{pmatrix} EZ_1\overline{Z_1} & EZ_1\overline{Z_2} \\ EZ_2\overline{Z_1} & EZ_2\overline{Z_2} \end{pmatrix} = \begin{pmatrix} s_1^2 & EZ_1\overline{Z_2} \\ EZ_2\overline{Z_1} & s_2^2 \end{pmatrix} = \begin{pmatrix} s_1^2 & r s_1 s_2 \\ \overline{r} s_1 s_2 & s_2^2 \end{pmatrix} \\
&\Lambda_Z^{-1} = \frac{1}{\left(1-|r|^2\right)s_1 s_2}\begin{pmatrix} \frac{s_2}{s_1} & -r \\ -\overline{r} & \frac{s_1}{s_2} \end{pmatrix} \\
&\det(\Lambda_Z) = \left(1-|r|^2\right)s_1^2 s_2^2 \ge 0 \\
&\left(\left(1-|r|^2\right)s_1 s_2\right)\underline{z}^{\dagger}\Lambda_Z^{-1}\underline{z} = \frac{s_2}{s_1}|z_1|^2 - 2\mathrm{Re}\left\{r\,\overline{z_1}z_2\right\} + \frac{s_1}{s_2}|z_2|^2
\end{aligned}}$$

- To see that this is true for n = 2 and $E\underline{Z} = 0$

  Bivariate Proper Gaussian Density

  Define

  - $Z_1 = X_1 + jY_1$, $Z_2 = X_2 + jY_2$ and $\underline{Z} = (Z_1, Z_2)^T$
  - $E\underline{Z} = 0$, so $EX_1 = EX_2 = EY_1 = EY_2 = 0$
  - $\underline{Z}$ is proper Gaussian, so $EZ_1Z_2 = 0$

    This gives

    $Cov(X_1, X_2) = Cov(Y_1, Y_2) \Rightarrow EX_1X_2 = EY_1Y_2$

    $Cov(X_i, X_i) = Cov(Y_i, Y_i) \Rightarrow s_{X_i}^2 = s_{Y_i}^2$

    $Cov(X_i, Y_j) = -Cov(Y_i, X_j) \Rightarrow EX_iY_j = -EY_iX_j$

    And $EX_iY_i = -EY_iX_i \Rightarrow EX_iY_i = 0$

  - $\boxed{s_1^2 = Var(Z_1) = E|Z_1|^2 = s_{X_1}^2 + s_{Y_1}^2 = 2s_{X_1}^2}$

    $\boxed{s_2^2 = Var(Z_1) = E|Z_2|^2 = s_{X_2}^2 + s_{Y_2}^2 = 2s_{X_2}^2}$

    So, we have $s_{X_1}^2 = s_{Y_1}^2 = \frac{1}{2}s_1^2$ and $s_{X_2}^2 = s_{Y_2}^2 = \frac{1}{2}s_2^2$

  - $r = \frac{1}{s_1 s_2}EZ_1\overline{Z_2}$

  a) $s_1^2 s_2^2 = \left(2s_{X_1}^2\right)\left(2s_{X_2}^2\right) \Rightarrow s_1 s_2 = 2s_{X_1}s_{X_2}$

  $$\begin{aligned}
  EZ_1\overline{Z_2} &= E\left(X_1 + jY_1\right)\left(X_2 - jY_2\right) \\
  &= E\left(X_1X_2 + Y_1Y_2 + jY_1X_2 - jX_1Y_2\right) \\
  &= E\left(X_1X_2 + X_1X_2 + jY_1X_2 + jY_1X_2\right) \\
  &= 2EX_1X_2 + 2jEY_1X_2
  \end{aligned}$$

$$r_{X_1,X_2} = \frac{Cov(X_1, X_2)}{s_{X_1} s_{X_2}} = \frac{E\left(X_1 - \cancel{EX_1}\right)\left(X_2 - \cancel{EX_2}\right)}{s_{X_1} s_{X_2}} = \frac{EX_1X_2}{s_{X_1} s_{X_2}}.$$

So, $EX_1X_2 = s_{X_1} s_{X_2} r_{X_1,X_2}$

$$r_{Y_1,X_2} = \frac{Cov(Y_1, X_2)}{s_{Y_1} s_{X_2}} = \frac{E\left(Y_1 - \cancel{EY_1}\right)\left(X_2 - \cancel{EX_2}\right)}{s_{Y_1} s_{X_2}} = \frac{EY_1X_2}{s_{Y_1} s_{X_2}} = \frac{EY_1X_2}{s_{X_1} s_{X_2}}.$$

So, $EY_1X_2 = s_{X_1} s_{X_2} r_{Y_1,X_2}$

Thus,

$$r = \frac{EZ_1\overline{Z_2}}{s_1 s_2} = \frac{\cancel{2}EX_1X_2 + \cancel{2}jEY_1X_2}{\cancel{2}s_{X_1} s_{X_2}}$$

$$= \frac{s_{X_1} s_{X_2} r_{X_1,X_2} + js_{X_1} s_{X_2} r_{Y_1,X_2}}{s_{X_1} s_{X_2}}$$

$$\boxed{r = r_{X_1,X_2} + j r_{Y_1,X_2}}$$

b) $\Lambda_Z = E\underline{Z}\underline{Z}^\dagger = E\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\begin{pmatrix} \overline{Z_1} & \overline{Z_2} \end{pmatrix} = \begin{pmatrix} EZ_1\overline{Z_1} & EZ_1\overline{Z_2} \\ EZ_2\overline{Z_1} & EZ_2\overline{Z_2} \end{pmatrix} = \begin{pmatrix} s_1^2 & EZ_1\overline{Z_2} \\ EZ_2\overline{Z_1} & s_2^2 \end{pmatrix}$

$$r = \frac{EZ_1\overline{Z_2}}{s_1 s_2} \quad \Rightarrow EZ_1\overline{Z_2} = r s_1 s_2, \quad EZ_2\overline{Z_1} = \overline{EZ_1\overline{Z_2}} = \overline{r} s_1 s_2$$

So, $\boxed{\Lambda_Z = \begin{pmatrix} s_1^2 & r s_1 s_2 \\ \overline{r} s_1 s_2 & s_2^2 \end{pmatrix}}$

$$\det(\Lambda_Z) = s_1^2 s_2^2 - r\,\overline{r}\, s_1^2 s_2^2$$

$$\boxed{\det(\Lambda_Z) = \left(1 - |r|^2\right) s_1^2 s_2^2}$$

Note that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Therefore,

$$\Lambda_Z^{-1} = \frac{1}{\left(1 - |r|^2\right) s_1^2 s_2^2}\begin{pmatrix} s_2^2 & -s_1 s_2 r \\ -s_1 s_2 \overline{r} & s_1^2 \end{pmatrix}$$

$$\boxed{\Lambda_Z^{-1} = \frac{1}{\left(1 - |r|^2\right) s_1 s_2}\begin{pmatrix} \dfrac{s_2}{s_1} & -r \\ -\overline{r} & \dfrac{s_1}{s_2} \end{pmatrix}}$$

c) Define $\underline{R} = \left(X_1, Y_1, X_2, Y_2\right)^T$

$$\Lambda_R = E\underline{R}\underline{R}^T = E\begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \end{pmatrix}\left(X_1, Y_1, X_2, Y_2\right) = \begin{pmatrix} EX_1X_1 & EX_1Y_1 & EX_1X_2 & EX_1Y_2 \\ EY_1X_1 & EY_1Y_1 & EY_1X_2 & EY_1Y_2 \\ EX_2X_1 & EX_2Y_1 & EX_2X_2 & EX_2Y_2 \\ EY_2X_1 & EY_2Y_1 & EY_2X_2 & EY_2Y_2 \end{pmatrix}$$

$$= \begin{pmatrix} s_{X_1}^2 & 0 & EX_1X_2 & -EY_1X_2 \\ 0 & s_{Y_1}^2 & EY_1X_2 & EX_1X_2 \\ EX_1X_2 & EY_1X_2 & s_{X_2}^2 & 0 \\ -EY_1X_2 & EX_1X_2 & 0 & s_{Y_2}^2 \end{pmatrix}$$

$$= \begin{pmatrix} s_{X_1}^2 & 0 & s_{X_1}s_{X_2}r_{X_1,X_2} & -s_{X_1}s_{X_2}r_{Y_1,X_2} \\ 0 & s_{Y_1}^2 & s_{X_1}s_{X_2}r_{Y_1,X_2} & s_{X_1}s_{X_2}r_{X_1,X_2} \\ s_{X_1}s_{X_2}r_{X_1,X_2} & s_{X_1}s_{X_2}r_{Y_1,X_2} & s_{X_2}^2 & 0 \\ -s_{X_1}s_{X_2}r_{Y_1,X_2} & s_{X_1}s_{X_2}r_{X_1,X_2} & 0 & s_{Y_2}^2 \end{pmatrix}$$

$$\boxed{\Lambda_R = \frac{1}{2}s_1s_2\begin{pmatrix} s_1/s_2 & 0 & r_{X_1,X_2} & -r_{Y_1,X_2} \\ 0 & s_1/s_2 & r_{Y_1,X_2} & r_{X_1,X_2} \\ r_{X_1,X_2} & r_{Y_1,X_2} & s_2/s_1 & 0 \\ -r_{Y_1,X_2} & r_{X_1,X_2} & 0 & s_2/s_1 \end{pmatrix}}$$

It can be shown that

$$\begin{pmatrix} a & 0 & c & -d \\ 0 & a & d & c \\ c & d & b & 0 \\ -d & c & 0 & b \end{pmatrix}^{-1} = \frac{1}{ab-\left(c^2+d^2\right)}\begin{pmatrix} b & 0 & -c & d \\ 0 & b & -d & -c \\ -c & -d & a & 0 \\ d & -c & 0 & a \end{pmatrix}$$

$$\boxed{\Lambda_R^{-1} = \frac{2}{\left(1-|r|^2\right)s_1s_2}\begin{pmatrix} s_2/s_1 & 0 & -r_{X_1,X_2} & r_{Y_1,X_2} \\ 0 & s_2/s_1 & -r_{Y_1,X_2} & -r_{X_1,X_2} \\ -r_{X_1,X_2} & -r_{Y_1,X_2} & s_1/s_2 & 0 \\ r_{Y_1,X_2} & -r_{X_1,X_2} & 0 & s_1/s_2 \end{pmatrix}}$$

It can also be shown that

$$\det\begin{pmatrix} a & 0 & c & -d \\ 0 & a & d & c \\ c & d & b & 0 \\ -d & c & 0 & b \end{pmatrix} = a^2b^2 - 2abd^2 - 2abc^2 + c^4 + 2c^2d^2 + d^4$$

$$= \left(ab - \left(c^2 + d^2\right)\right)^2$$

Thus, $\det\left(\Lambda_R\right) = \left(\frac{1}{2}s_1s_2\right)^4\left(1-|r|^2\right)^2$

$$\boxed{\det\left(\Lambda_R\right)=\frac{s_1^4 s_2^4}{16}\left(1-|r|^2\right)^2}$$

d) $\sqrt{\det\left(\Lambda_R\right)}=\frac{s_1^2 s_2^2}{4}\left(1-|r|^2\right)$

From part (b) we have $\det\left(\Lambda_Z\right)=\left(1-|r|^2\right)s_1^2 s_2^2$

Therefore, $\sqrt{\det\left(\Lambda_R\right)}=\frac{\det\left(\Lambda_Z\right)}{4}$

e) We want to show that $f_{X_1,Y_1,X_2,Y_2}\left(x_1,y_1,x_2,y_2\right)=f_{Z_1,Z_2}\left(z_1,z_2\right)$ where

$$f_{Z_1,Z_2}\left(z_1,z_2\right)=\frac{1}{p^2\det\left(\Lambda_Z\right)}e^{-(\underline{z})^\dagger\Lambda_Z^{-1}(\underline{z})}$$

$$f_{X_1,Y_1,X_2,Y_2}\left(x_1,y_1,x_2,y_2\right)=\frac{1}{(2p)^2\sqrt{\det\left(\Lambda_R\right)}}e^{-\frac{1}{2}\underline{r}^T\Lambda_R^{-1}\underline{r}}$$

$$\underline{z}=\left(z_1,z_2\right)^T=\left(x_1+jy_1,x_2+jy_2\right)^T$$

$$\underline{r}=\left(x_1,y_1,x_2,y_2\right)^T$$

I. First, compare $p^2\det\left(\Lambda_Z\right)$ and $(2p)^2\sqrt{\det\left(\Lambda_R\right)}$

$$(2p)^2\sqrt{\det\left(\Lambda_R\right)}=4p^2\sqrt{\det\left(\Lambda_R\right)}=4p^2\frac{\det\left(\Lambda_Z\right)}{4}=p^2\det\left(\Lambda_Z\right)$$

II. Next, compare $\underline{z}^\dagger\Lambda_Z^{-1}\underline{z}$ and $\frac{1}{2}\underline{r}^T\Lambda_R^{-1}\underline{r}$

$$\left(\left(1-|r|^2\right)s_1 s_2\right)\underline{z}^\dagger\Lambda_Z^{-1}\underline{z}=\left(\overline{z_1},\overline{z_2}\right)\begin{pmatrix}\frac{s_2}{s_1} & -r\\ -\overline{r} & \frac{s_1}{s_2}\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix}=\frac{s_2}{s_1}\overline{z_1}z_1-r\overline{z_1}z_2-\overline{r}\,\overline{z_2}z_1+\frac{s_1}{s_2}\overline{z_2}z_2$$

$$=\frac{s_2}{s_1}|z_1|^2-2\mathrm{Re}\left\{r\overline{z_1}z_2\right\}+\frac{s_1}{s_2}|z_2|^2$$

$$\begin{aligned}\mathrm{Re}\left\{r\overline{z_1}z_2\right\}&=\mathrm{Re}\left\{\left(r_{X_1,X_2}+j\,r_{Y_1,X_2}\right)\left(x_1-jy_1\right)\left(x_2+jy_2\right)\right\}\\&=r_{X_1,X_2}x_1x_2+r_{X_1,X_2}y_1y_2-r_{Y_1,X_2}x_1y_2+r_{Y_1,X_2}y_1x_2\\&=r_{X_1,X_2}\left(x_1x_2+y_1y_2\right)+r_{Y_1,X_2}\left(y_1x_2-r_{Y_1,X_2}\right)\end{aligned}$$

$$\left(1-|r|^2\right)s_1 s_2\left(\frac{1}{2}\underline{r}^T\Lambda_R^{-1}\underline{r}\right)=\left(x_1,y_1,x_2,y_2\right)\begin{pmatrix} s_2/s_1 & 0 & -r_{X_1,X_2} & r_{Y_1,X_2} \\ 0 & s_2/s_1 & -r_{Y_1,X_2} & -r_{X_1,X_2} \\ -r_{X_1,X_2} & -r_{Y_1,X_2} & s_1/s_2 & 0 \\ r_{Y_1,X_2} & -r_{X_1,X_2} & 0 & s_1/s_2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

$$=\left(x_1^2+y_1^2\right)\frac{s_2}{s_1}+\left(x_2^2+y_2^2\right)\frac{s_1}{s_2}$$

$$-2\left(\left(y_1y_2+x_1x_2\right)r_{X_1,X_2}+\left(y_1x_2-x_1y_2\right)r_{Y_1,X_2}\right)$$

$$=|z_1|^2\frac{s_2}{s_1}+|z_2|^2\frac{s_1}{s_2}$$

$$-2\left(\left(y_1y_2+x_1x_2\right)r_{X_1,X_2}+\left(y_1x_2-x_1y_2\right)r_{Y_1,X_2}\right)$$

$$=|z_1|^2\frac{s_2}{s_1}+|z_2|^2\frac{s_1}{s_2}-2\mathrm{Re}\left\{r\,\overline{z_1}z_2\right\}$$

Therefore, $\left(\left(1-|r|^2\right)s_1 s_2\right)\underline{z}^\dagger\Lambda_Z^{-1}\underline{z} = \left(1-|r|^2\right)s_1 s_2\left(\frac{1}{2}\underline{r}^T\Lambda_R^{-1}\underline{r}\right)$.

And $\underline{z}^\dagger\Lambda_Z^{-1}\underline{z} = \frac{1}{2}\underline{r}^T\Lambda_R^{-1}\underline{r}$

From I and II, we have $\frac{1}{p^2\det\left(\Lambda_Z\right)}e^{-(\underline{z})^\dagger\Lambda_Z^{-1}(\underline{z})} = \frac{1}{(2p)^2\sqrt{\det\left(\Lambda_R\right)}}e^{-\frac{1}{2}\underline{r}^T\Lambda_R^{-1}\underline{r}}$.

Hence, $\boxed{f_{X_1,Y_1,X_2,Y_2}\left(x_1,y_1,x_2,y_2\right)=\frac{1}{p^2\det\left(\Lambda_Z\right)}e^{-(\underline{z})^\dagger\Lambda_Z^{-1}(\underline{z})}}$ QED

- Proper Gaussian is preserved under linear filter:

$$\text{complex } z(t) \longrightarrow \boxed{\begin{matrix}\text{complex}\\ h(s,t)\\ \text{Linear}\end{matrix}} \longrightarrow \text{complex } w(t)$$

$w(t)=\int h(s,t)z(s)ds$

If $\{Z(t)\}$ is proper Gaussian, so is $\{w(t)\}$

- If $\{Z(t)\}$ is 0-mean proper Gaussian, so is $\{w(t)\}$

$$EW(u)W(v)=E\int h(s,u)z(s)ds\int h(s',v)z(s')ds'$$

$$=E\iint h(s,u)h(s',v)z(s)z(s')dsds'$$

$$=\iint h(s,u)h(s',v)\underbrace{\left(E\left(z(s)z(s')\right)\right)}_{0}dsds'$$

$$=0$$

- For non-zero mean case, we have

$$E\left(z(s)z(s')\right)-Ez(s)Ez(s') = 0$$

$$EW(u)=\int h(s,u)Ez(s)ds$$

$$EW(v)=\int h(s',v)Ez(s')ds'$$

$$EW(u)W(v)=\iint h(s,u)h(s',v)\left(E\left(z(s)z(s')\right)\right)dsds'$$ as before.

$$EW(u)EW(v)=\int h(s,u)Ez(s)ds\int h(s',v)Ez(s')ds'$$

$$=\iint h(s,u)h(s',v)\left(Ez(s)Ez(s')\right)dsds'$$

$$EW(u)W(v)-EW(u)EW(v)$$

$$=\iint h(s,u)h(s',v)\underbrace{\left(E\left(z(s)z(s')\right)-Ez(s)Ez(s')\right)}_{0}dsds'$$

$$=0$$