Review

- Suppose $A_1, A_2, \ldots$ is a countable collection of sets. The Cartesian product $A_1 \times A_2 \times \cdots$ is defined to be the set of sequences $(a_1, a_2, \ldots)$ where each a_n belongs to A_n.
  The **countable axiom of choice** asserts that if the sets A_n are all non-empty, then the Cartesian product is also non-empty.
- If a, b, and c are non-negative real numbers, such that $a \le b + c$, then $\frac{a}{1+a} \le \frac{b}{1+b} + \frac{c}{1+c}$. (Converse is false, consider $b = c = 1$, and $a = 3$).
  $$\frac{d}{dx}\frac{x}{1+x} = \frac{1}{(1+x)^2} > 0. \quad \frac{a}{1+a} \le \frac{b+c}{1+b+c} \le \frac{b}{1+b+c} + \frac{c}{1+b+c} \le \frac{b}{1+b} + \frac{c}{1+c}.$$
- If p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ (in particular, $p > 1$ and $q > 1$), then
  - $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for all nonnegative real numbers a and b.
  - Hölder's inequality: $\forall x, y \in \mathbb{C}^n$, $\left|\sum_{k=1}^{n} x_k y_k\right| \le \|x\|_p \|y\|_q$.
- Minkowski's inequality: $\|x + y\|_p \le \|x\|_p + \|y\|_p$, $p \ge 1$.

Euclidean Space and Metric Spaces

9.1 Structures on Euclidean Space

- Convention:
  - Letters at the end of the alphabet $\vec{x}, \vec{y}, \vec{z}$, etc., will be used to denote points in $\mathbb{R}^n$, so $\vec{x} = (x_1, x_2, \ldots, x_n)$ and x_k will always refer to the k^{th} coordinate of $\vec{x}$.
- Def: $\mathbb{R}^n$ is the set of ordered n-tuples $\vec{x} = (x_1, x_2, \ldots, x_n)$ of real numbers.
- The **vector space** axioms:
  A set V with a vector addition and scalar multiplication is said to be a vector space over the scalar field $(\mathbb{R} \text{ or } \mathbb{C})$ provided
  1. vector addition satisfies the commutative group axioms:
     - commutativity: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
     - associativity: $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$,
     - existence of zero: $\vec{x} + \vec{0} = \vec{0}\ \forall \vec{x}$, and

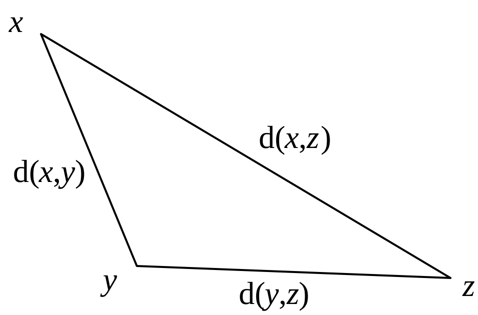
- existence of additive inverses: $\vec{x} + (-\vec{x}) = \vec{0}$; and

2. scalar multiplication
   - is associative: $(ab)\vec{x} = a(b\vec{x})$ and
   - distributes over addition in both ways: $a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$ and $(a+b)\vec{x} = a\vec{x} + b\vec{x}$

- **Metric space** ***(M,d)***

  Def: A metric space M is a set with a real-valued **distance function** (or **metric**) $d(x,y): M \times M \to \mathbb{R}$ defined for x, y in M satisfying

  1) positivity: $d(x,y) \geq 0$ with equality if and only if $x = y$,
     - $\forall x\ d(x,x) = 0$. $x \neq 0 \Rightarrow d(x,y) > 0$.
  2) symmetry: $d(x,y) = d(y,x)$,
  3) triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$.



- There is no need to assume that the space has a vector space structure.
- If (M,d) is a metric space, then

  $M_1 \subset M \Rightarrow (M_1, d)$ is also a metric space.
- $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{n-1}, x_n)$
- "Quadrilateral inequality": $|d(x,y) - d(u,v)| \leq d(x,u) + d(y,v)$

  $$d(x,y) \leq d(x,u) + d(u,v) + d(y,v) \Leftrightarrow d(x,y) - d(u,v) \leq d(x,u) + d(y,v)$$

  $$d(u,v) \leq d(u,x) + d(x,y) + d(y,v) \Leftrightarrow d(u,v) - d(x,y) \leq d(x,u) + d(y,v)$$

  - $|d(x,y) - d(y,z)| \leq d(x,z)$

    $$d(x,y) \leq d(x,z) + d(y,z) \Leftrightarrow d(x,z) \geq d(x,y) - d(y,z)$$

    $$d(y,z) \leq d(y,x) + d(x,z) \Leftrightarrow d(x,z) \geq d(y,z) - d(x,y)$$

- $\boxed{|d(x,y) - d(y,z)| \leq d(x,z) \leq d(x,y) + d(y,z)}$

- Example of metric.
  - **Euclidean (Pythagorean) distance** between $\bar{x}$ and $\bar{y}$: $d(\bar{x},\bar{y}) = \sqrt{\sum_{j=1}^{n}\left(x_j - y_j\right)^2}$.
    - $\mathbb{R}^n$ with Pythagorean distance functions forms a metric space.
  - If (M,d) is a metric space, then (M,d_1) where $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ is also a metric space.
  - For any non-empty set *M*. The discrete metric $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$. (*M*,*d*) is a metric space.
  - $d(x,y) = d^N\left(f(x), f(y)\right)$. $f : M \xrightarrow{1:1} N$. and d^N is a metric on *N*.

    $d(x,y) = d^N\left(f(x), f(y)\right) \geq 0$. $d(x,y) = d(y,x)$.

    $d(x,x) = d^N\left(f(x), f(x)\right) = 0$

    $d(x,y) = 0 \Leftrightarrow d^N\left(f(x), f(y)\right) = 0 \Leftrightarrow f(x) = f(y) \Leftrightarrow x = y$

    $$\begin{aligned} d(x,z) &= d^N\left(f(x), f(z)\right) \leq d^N\left(f(x), f(y)\right) + d^N\left(f(y), f(z)\right). \\ &= d(x,y) + d(y,z) \end{aligned}$$
  - $d(x,y) = \left|f(x) - f(y)\right|$, where *f* is one-to-one: $M \to \mathbb{R}$.

- Def: A **norm** on a real or complex vector space is a function $\|\bar{x}\|$ defined for every $\bar{x}$ in the vector space satisfying
  1) positivity: $\|\bar{x}\| \geq 0$ with equality if and only if $\bar{x} = 0$,
  2) homogeneity: $\|a\bar{x}\| = |a|\|\bar{x}\|$ for any scalar *a*,
  3) triangle inequality: $\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$.

  - A norm must be defined on a vector space in order for conditions 2 and 3 to make sense.
  - $\left|\|\bar{x}\| - \|\bar{y}\|\right| \leq \|\bar{x} - \bar{y}\|$.

    Proof. $\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$. Let $\bar{x} = \bar{z} - \bar{y}$. Then $\|\bar{z}\| - \|\bar{y}\| \leq \|\bar{z} - \bar{y}\|$. Switching $\bar{y}$ and $\bar{z}$, we have $\|\bar{y}\| - \|\bar{z}\| \leq \|\bar{y} - \bar{z}\| = \|\bar{z} - \bar{y}\|$.
- The absolute value and the norm coincide for $\mathbb{R}^1$
- Use single bars for the norm on $\mathbb{R}^n$.
- Examples of norms on $\mathbb{R}^n$

- (Minkowski) **p-norm**: $\|\vec{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$ where p is a constant satisfying $1 \le p < \infty$.
  - Def: A **Euclidean norm** on $\mathbb{R}^n$ is a function $|\vec{x}| = \|\vec{x}\|_2 = \sqrt{\sum_{j=1}^{n} x_j^2}$ defined for every $\vec{x}$ in $\mathbb{R}^n$.
    - Euclidean norm $|\vec{x}|$ is a norm $\Rightarrow$ $|\vec{x} - \vec{y}|$ is a metric.
  - $\|\vec{x}\|_1 = \sum_{j=1}^{n} |x_j|$
    - Interpret the distance $\|\vec{x} - \vec{y}\|_1$ as the shortest distance between $\vec{x}$ and $\vec{y}$ along a broken line segment that moves parallel to the axes.
  - $\|\vec{x}\|_{\sup} = \max_j \{|x_j|\} = \|\vec{x}\|_\infty = \lim_{p \to \infty} \|\vec{x}\|_p$
- Ex Let $C([a,b])$ denote the continuous functions on $[a,b]$.

  Then, $\|f\|_{\sup} = \sup_x |f(x)|$ is a norm on $C([a,b])$, called the sub norm.
- If $\|\vec{x}\|$ is a norm, then $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$ (called the induced metric) is a metric.
  - The metric $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$ is said to be the metric associated with (or induced by) the norm.
- If $\|x\|$ is any norm on $\mathbb{R}^n$, then there exists a positive constant M such that $\forall x \in \mathbb{R}^n$, $\|x\| \le M|x|$. One possible M is $\sqrt{\sum_{j=1}^{n} \|e^{(j)}\|^2}$.
- Let $f(x) = \|x\| : M \to \mathbb{R}$. f is continuous if $d^{\mathbb{R}}$ is associated to a norm (any norm in $\mathbb{R}$) and one of these occurs

  (1) $d^M(y,x) = \|y - x\|$.

  (2) $M \subset \mathbb{R}^n$. d^M is associated to a norm (any norm in $\mathbb{R}^n$).

  Proof. (1) $\|f(y) - f(x)\| \le M_1 |f(y) - f(x)| = M_1 \big| \|y\| - \|x\| \big|$

  $$\le M_1 \|y - x\| \le M_1 M_2 |y - x|$$

  (2) $d^{\mathbb{R}}(f(y), f(x)) \le M_1 \|y - x\| \overset{(b)}{\le} M_1 M_2 \|y - x\|'$ where $\|x\|'$ is a norm on $M \subset \mathbb{R}^n$ with which d^M is associated.

  (a) and (b): Any norm on $\mathbb{R}^n$ are equivalent.

- $f(x) = \|x\| : M \subset \mathbb{R}^n \to \mathbb{R}$ is continuous when the metrics d^M and $d^{\mathbb{R}}$ are Euclidean.

- Def: An **inner product on a real vector space** is a real-valued function $\langle \vec{x}, \vec{y} \rangle$ defined for all $\vec{x}$ and $\vec{y}$ in the vector space satisfying
  1) symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$,
  2) bilinearity: $\langle a\vec{x} + b\vec{y}, \vec{z} \rangle = a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle$ and $\langle \vec{x}, a\vec{y} + b\vec{z} \rangle = a\langle \vec{x}, \vec{y} \rangle + b\langle \vec{x}, \vec{z} \rangle$ for all real numbers *a*, *b*,
  3) positive definiteness: $\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality if and only if $\vec{x} = \vec{0}$.

- **Cauchy-Schwartz Inequality**

  On a real or complex inner product space, $\left|\langle \vec{x}, \vec{y} \rangle\right| \leq \sqrt{\langle \vec{x}, \vec{x} \rangle}\sqrt{\langle \vec{y}, \vec{y} \rangle}$, with equality if and only if $\vec{x}$ and $\vec{y}$ are collinear.

- If $\langle \vec{x}, \vec{y} \rangle$ is an inner product, then $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ is the associated or induced norm.
  - which implies $\|\vec{x} - \vec{y}\|$ is a metric.
  - Not every norm is associated to an inner product. (Among $\|\vec{x}\|_1$, $\|\vec{x}\|_2$, $\|\vec{x}\|_{\sup}$, only $\|\vec{x}\|_2$ is.)

- An inner product defines a norm via $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$, and a norm defines a metric via $d(x, y) = \|x - y\|$.

- Ex On $\mathbb{R}^n$, the scalar product or dot product $\vec{x} \cdot \vec{y} = \sum_{j=1}^{n} x_j y_j$ is an inner product;

  hence, $|\vec{x}| = \sqrt{\sum_{j=1}^{n} x_j^2}$ is a norm and $|\vec{x} - \vec{y}|$ is a metric.

- On an inner product space,
  - the polarization identity $\langle \vec{x}, \vec{y} \rangle = \frac{1}{4}\left(\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2\right)$ holds. $\left(\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}\right)$
  - the associated norm satisfies the **parallelogram law** $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2$.
    - Geometrically, the parallelogram law can be interpreted to say the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides.

- If a norm $\|\cdot\|$ satisfies the parallelogram law $\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2$, then the polarization identity $g(x,y)=\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right)$ defines an inner product.

- The norm associated with this inner product $f(x)=\sqrt{g(x,x)}$ is the original norm $\|\cdot\|$.

- If a norm $\|\cdot\|$ satisfies the parallelogram law $\|\vec{x}+\vec{y}\|^2+\|\vec{x}-\vec{y}\|^2=2\|\vec{x}\|^2+2\|\vec{y}\|^2$, then it is induced by an inner product $\langle x,y\rangle=\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right)$.
- $\mathbb{C}^n$ is the set of n-tuples $\vec{z}=(z_1,z_2,\ldots,z_n)$ of complex numbers.
  - Has complex dimension n since the basis vectors $\hat{e}^{(1)},\ldots,\hat{e}^{(n)}$ of $\mathbb{R}^n$ also form a basis of $\mathbb{C}^n$, $\vec{z}=z_1\hat{e}^{(1)}+\cdots+z_n\hat{e}^{(n)}$.
  - Regarded as a real vector space, $\mathbb{C}^n$ has dimension $2n$ with $\hat{e}^{(1)},\ldots,\hat{e}^{(n)},i\hat{e}^{(1)},\ldots,i\hat{e}^{(n)}$ forming a basis.
- Def: A complex inner product on a complex vector space is a complex-valued function $\langle\vec{x},\vec{y}\rangle$ defined for all $\vec{x}$ and $\vec{y}$ in the space satisfying

  1) Hermitian symmetry: $\langle x,y\rangle=\overline{\langle y,x\rangle}$,
  2) Hermitian linearity: $\langle a\vec{x}+b\vec{y},\vec{z}\rangle=a\langle\vec{x},\vec{z}\rangle+b\langle\vec{y},\vec{z}\rangle$ and $\langle\vec{x},a\vec{y}+b\vec{z}\rangle=\overline{a}\langle\vec{x},\vec{y}\rangle+\overline{b}\langle\vec{x},\vec{z}\rangle$,
  3) positive definiteness: $\langle\vec{x},\vec{x}\rangle$ is real and $\langle\vec{x},\vec{x}\rangle\geq 0$ with equality if and only if $\vec{x}=\vec{0}$.

- For $\mathbb{C}^n$, the usual inner product is $\langle\vec{z},\vec{w}\rangle=\sum_{j=1}^{n}z_j\overline{w}_j$.
- Let $\langle\cdot,\cdot\rangle$ denotes an inner product on a (real or complex) vector space V and let $\|\cdot\|$ be the corresponding norm. Then
  - $\langle x,y\rangle=\langle x,z\rangle\ \forall x\in V\Rightarrow y=z$
  - $\langle\cdot,\cdot\rangle$ is continuous
  - Pythagorus identity: $\langle x,y\rangle=0\Rightarrow\|x+y\|^2=\|x\|^2+\|y\|^2$
  - Parallelogram law: $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$
  - Polarization identity:

    Real case: $\langle x,y\rangle=\frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right)$.

Complex case: $\langle z,w\rangle = \frac{1}{4}\left(\|z+w\|^2 - \|z-w\|^2 + i\|z+iw\|^2 - i\|z-iw\|^2\right)$.

- Cauchy-Schwarz inequality: $|\langle x,y\rangle| \le \|x\|\|y\|$

9.2 Topology of Metric Spaces

- (M,d) is a metric space. $M' \subset M \Rightarrow (M',d)$ is also a metric space.
- Def: A **subspace** M' of a metric space M is a subset of M with the same metric.
- Use the word "ball" for the solid region, and "sphere" for the boundary.
- Def: The **open ball** $B_r(y)$ in a metric space M with center y and radius r is $B_r(y) = \{x \in M : d(x,y) < r\}$.
  - If $U \subset W$, the open ball in U are the intersections of U with open balls in W with the same center and radius: $B_r^U(y) = U \cap B_r^W(y)$.
  - If $U \subset W$ and U is open in W, then $\forall y \in U\ \exists r$ (small enough, depending on y) such that $B_r(y)_U = B_r(y)_W$.
  - An open ball $B_r^M(y)$ also contains open balls $B_{r-d(x,y)}^M(x)$ centered at all its other points $x \in B_r^M(y)$. $\Rightarrow$ open ball in M is an open set in M.
- Def: A subset A of a metric space M is said to be **open** in M if
  - every point of A lies in an open ball (in M) entirely contained in A.

  $\equiv$ $\forall x \in A\ \exists r_x > 0$ such that $B_r^M(x) \subset A$.

  $\equiv$ $\forall x \in A\ \exists r_x > 0$ such that $\forall y \in M\ \ d(y,x) < r \Rightarrow y \in A$

  $\equiv$ $\forall x \in A\ \exists r_x > 0$ such that $(M \setminus A) \cap B_r^M(x) = \varnothing$.

  $\Leftrightarrow$ $\mathring{A} = A$.

  $\Leftrightarrow$ $M \setminus A$ is closed in M.
- $A \subset M$ is not open in M iff
  - $\exists x \in A$ such that $\forall r > 0\ \ B_r^M(x) \not\subset A$

  $\equiv$ $\exists x \in A$ such that $\forall r > 0\ \ (M \setminus A) \cap B_r^M(x) \ne \varnothing$
- Let M be a matrix subspace of a metric space M_1. $(M \subset M_1)$.

  Then, for $A \subset M$
  - A is open in M if and only if there exists an open subset A_1 of M_1 such that $A = A_1 \cap M$.
  - If M is open in M_1, then A is open in M if and only if A is open in M_1.

- Let M_{small} be a matrix subspace of a metric space M_{big} . $\left(M_{small} \subset M_{big}\right)$.

  Then, for $A_{small} \subset M_{small}$,

  - A_{small} is **open in** M_{small} if and only if there exists an open subset A_{big} of M_{big} such that $A_{small} = A_{big} \cap M_{small}$.
  - If M_{small} is **open in** M_{big}, then A is open in M_{small} if and only if A is open in M_{big}.

- $A \subset M$. B is open in $M \Rightarrow B \cap A$ is open in A.
- Examples of open set in M
  - $\varnothing$, M.
  - (0,1) open in $\mathbb{R}$. $\{0\} \times (0,1)$ is not open in $\mathbb{R}^2$.

- Theorem: In any metric space,
  - an **arbitrary union of open sets** is open.
  - a **finite intersection of open sets** is open.

- Def: A **neighborhood** of a point is an open set containing the point.
- Def: The **interior** of a set A is the set of all points contained in open balls contained in A.

  $\overset{\circ}{A} = \left\{ x \in A; \exists r_x > 0 \text{ such that } B_{r_x}^M(x) \subset A \right\}$

- **Sequence**: $\{x_n\}$: $x_1, x_2, \ldots$
  - **Range** of $\{x_n\}$ is the set of all points x_n (n = 1, 2, 3, …). May be finite or infinite).
  - **The sequence is bounded if its range is bounded.**

- **Convergence, limit of a sequence**
  - Def: If $x_1, x_2, \ldots$ is a sequence of points in M, then the sequence has a **limit** x (in M) (or the sequence **converge**s (in M) to x), written $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$, provided that
    - $\forall \frac{1}{m}\ \exists N$ such that $\forall n \geq N\ \ d(x_n, x) \leq \frac{1}{m}$.

    $\equiv$ $\lim\limits_{n\to\infty} d(x_n, x) = 0$ in the Euclidean sense.

    $\equiv$ every neighborhood of x contains all but a finite number of x_n.
  - $\lim\limits_{n\to\infty} x_n = x \Leftrightarrow \lim\limits_{n\to\infty} d(x_n, x) = 0$ in the Euclidean sense.
  - Definition of "convergent sequence" depends not only on $\{x_n\}$ but also on M.

- Let $x, x' \in M$. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = x'$, then $x = x'$.

  Proof. $\forall e$ take N large enough. Then $\forall n \geq N$, both $d(x_n, x)$ and $d(x_n, x')$ are $\leq \frac{e}{2}$. So, $d(x, x') \leq d(x_n, x) + d(x_n, x') \leq e$.

- If $\{x_n\}$ converges, then $\{x_n\}$ is bounded..

  Proof. Let $\lim_{n\to\infty} x_n = x$. Then $\exists N \ \forall n \geq N \ \ d(x_n, x) \leq 1$. Let $r = \max\{1, d(x_1, x), d(x_2, x), \ldots, d(x_N, x)\}$. Then $d(x_n, x) \leq r$ for all *n*.

- Def: If $\{x_n\}$ does not converge, it is **diverge**.

- On $C([a,b])$,

  convergence in the sup-norm metric $\left(\forall \frac{1}{m} \exists N \ \forall k \geq N, \sup_x \left| f_k(x) - f(x) \right| \leq \frac{1}{m}\right)$

  is the same as

  uniform convergence $\left(\forall \frac{1}{m} \exists N \ \forall k \geq N \ \forall x \left| f_k(x) - f(x) \right| \leq \frac{1}{m}\right)$.

  $\sup_x \left| f_k(x) - f(x) \right| \leq \frac{1}{m} \Leftrightarrow \forall x \left| f_k(x) - f(x) \right| \leq \frac{1}{m}$.

- If $x_n \to x$ in a metric space and *y* is any other point in the space, then $\lim_{n\to\infty} d(x_n, y) = d(x, y)$ in the Euclidean sense.

- If $x_n \to x$ and $y_n \to y$ in a metric space, then $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ in the Euclidean sense.

  Use quadrilateral inequality: $\left| d(x, y) - d(x_n, y_n) \right| \leq d(x, x_n) + d(y, y_n)$.

- $\mathbb{R}^n$ and Euclidean metric.

  - A sequence $x^{(1)}, x^{(2)}, \ldots$ in $\mathbb{R}^n$ converges to *x* if and only if the sequence of coordinates $x_k^{(1)}, x_k^{(2)}, \ldots$ converges to x_k for every *k* = 1, …, *n*

  Proof. "$\Rightarrow$": $\forall e \ \exists N \ \forall n \geq N$

  $\left| (x_n)_k - (y)_k \right| \leq \sqrt{\sum_{k=1}^{K} \left| (x_n)_k - (y)_k \right|^2} = |x - y| \leq e$. "$\Leftarrow$": $\forall e \ \exists N \ \forall n \geq N$

  $\left| (x_n)_k - (y)_k \right| \leq \frac{e}{\sqrt{K}}$. $|x - y| = \sqrt{\sum_{k=1}^{K} \left| (x_n)_k - (y)_k \right|^2} \leq \sqrt{K \left(\frac{e}{\sqrt{K}} \right)^2} = e$.

- Suppose $\{x_n\}$ and $\{y_n\}$ are sequence in $\mathbb{R}^k$, $\{a_n\}$ is a sequence of real numbers, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} a_n = a$. Then (*a*) $\lim_{n\to\infty}(x_n + y_n) = x + y$, (*b*) $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$, (*c*) $\lim_{n\to\infty}(a_n x_n) = ax$.

  Proof. Convergence of $\{x_n\}$ and $\{y_n\}$ implies convergence of all their component. Consider the above operations for each component, then, from what we know about sequence in $\mathbb{R}$, we know that they converges for each component. Because all component converges, this prove (*a*) and (*c*). For (*b*), we know that finite addition of convergent sequences in $\mathbb{R}$ converges.

- Def: x is a **limit point of a sequence** $\{x_n\}$ if
  - every neighborhood of x contains x_n for infinitely many n.
  - ≡ There exists a subsequence (x_{n_k}) such that $x_{n_k} \to x$.

- **Limit point of a set**
  - Def: x ($\in M$) is a **limit point of a set** A ~~$(\subset M)$~~
    - if every neighborhood of x contains points of A not equal to x.
    - ≡ $\forall r > 0 \ \exists y \in B_r^A(x)$ such that $y \neq x$.
    - ≡ There exists a sequence of point $\neq x$ in A converging to x.
    - ≡ Every neighborhood of x contains infinitely many points of A.

      Proof. 1) "⇐" for any neighborhood, has infinite point of A; so at least one point is not x. 2) "⇒" Assume a neighborhood contain only finite points $\neq x$ of A. Then, there exists min distance r to x, inside which no points in A except may be x.
    - ⇒ There exists a sequence of point in A converging to x.

      Proof. Pick sequence $x_n \in B_{\frac{1}{n}}^A(x)$.
  - If $\exists r > 0$ such that $B_r^M(x) \cap A = \varnothing$, then x is not a limit point of A.
  - A finite set has no limit point.

    Proof. Need every neighborhood of the limit point to contain infinitely many points of the set.
  - If $x \in M$ is a limit point of a set A. Then, $x \in M$ is a limit point of a set $B \supset A$.

    Proof $\exists y \in B_r^A(x) \subset B_r^B(x)$ such that $y \neq x$.
  - Every point of an open set is a limit-point.

- Def: A set is **closed** in M
  - if it contains all its limit points.

$\Leftrightarrow$ $\bar{A} = A$

$\Leftrightarrow$ $M \setminus A$ is open in M.

$\equiv$ Whenever the terms of a convergent sequence are in A, the limit must also be in A.

- Example of closed sets
  - A set with no limit points such as the empty set, or a finite set, is automatically closed.
  - Closed ball in M with center $y \in M$ and radius r, $A = \{x \in M : d(x, y) \le r\}$.

    Proof. Let x be a limit point of A. Then, there exists a sequence $(x_n)_{n=1}^{\infty} \subset A$ converging to x. Because $\forall n\, d(x_n, y) \le r$, $d(x, y) = \lim_{n\to\infty} d(x_n, y) \le r$.

  - Sphere in M with center $y \in M$ and radius r, $A = \{x \in M : d(x, y) = r\}$

    Proof. $\forall n\, d(x_n, y) = r$. Thus, $d(x, y) = \lim_{n\to\infty} d(x_n, y) = r$.

- Def: The **closure** of a set consists of the set together with all its limit points. $\bar{A} = A \cup \{\text{limit points of } A\}$.
- The closure is always a closed set.
  - x is a limit point of closure of $A \Rightarrow x$ is a limit point of A.
- A set is closed if and only if it equals its closure.
- Def: If $A \subset B$, A is **dense** in B (A is a dense subset of B) if
  - the closure of A contains B. $\left(A \subset B \subset \text{closure}(A)\right)$.

  $\equiv$ Every point in B is either a point of A or a limit-point of A.
- In a metric space, **a set is closed if and only if its complement is open.**

- **Finite unions and arbitrary intersections of closed sets** are closed.

- **Cauchy sequence**
  - Def: $\{x_n\}$ is a Cauchy sequence if $\forall \frac{1}{m}$ $\exists N$ such that $\forall j, k \ge N$, $d(x_j, x_k) \le \frac{1}{m}$.
  - A convergent sequence is always a Cauchy sequence $\left(d(x_j, x_k) \le d(x, x_k) + d(x, x_k)\right)$.
    - The converse is not true for the general metric space.
      - Ex. rational numbers
  - Let $\{x_n\}$ be a Cauchy sequence. If there exists a subsequence converging to x, then the whole sequence converges to x. (Consider sequence $x_1, x, x_2, x, \ldots$)
  - On $C([a,b])$, the Cauchy criterion for a sequence $\{f_n\}$ in the sup-norm metric is identical to the uniform Cauchy criterion.

$$\left(\sup_x \left| f_j(x) - f_k(x) \right| \le \frac{1}{m} \Leftrightarrow \forall x \left| f_j(x) - f_k(x) \right| \le \frac{1}{m}\right).$$

- **complete**
  - Def: A metric space is complete
    - if every Cauchy sequence has a limit.
    - if every Cauchy sequence is convergent.
    - Ex. $\mathbb{R}^n$, $C([a,b])$ with the sup-norm metric
    - Ex *not complete:* $C([a,b])$ with the L^1 metric: $d(f,g) = \int_a^b |f(x) - g(x)| dx$.

      (consider a sequence of continuous functions converging Pointwise to a discontinuous function.)
    - Ex. M not complete: $(\mathbb{R}, d)$, $d(x,y) = |f(x) - f(y)|$, where $f(x) = \arctan(x), e^x$, or $\frac{1}{x}$ with $(\mathbb{R}^+ / \{0\}, d)$ or some other one-to-one function whose tail converges to some value but never reach that value.

      Take $\{x_n\}$ ($\{n\}$ or $\{-n\}$) to be a sequence going along the tail direction. Then, $\lim_{n\to\infty} f(x_n) = a$. Sequence $\{x_n\}$ is Cauchy because

      $$\begin{aligned} d(x_n, x_m) &= |f(x_n) - f(x_m)| = |f(x_n) - a + a - f(x_m)| \\ &\le |f(x_n) - a| + |a - f(x_m)| \end{aligned}$$

      and $|f(x_n) - a|$, $|a - f(x_m)|$ can be made < any e by taking n, m big enough.

      Assume $\exists x_0 \in M$ such that $n \xrightarrow{d} x_0$. This means $d(n, x_0) \xrightarrow{\text{Euclidean}} 0$. So, $\lim_{n\to\infty} d(n, x_0) = \lim_{n\to\infty} |f(n) - f(x_0)| = 0$. Now, we know that the sequence $x_n = f(n) \to a$ in Euclidean $(d_2(x,y) = |x - y|)$. So, have $\lim_{n\to\infty} d_2(x_n, f(x_0)) = d_2(a, f(x_0)) = 0$. Contradiction because there is no $x_0 \in M$ such that $f(x_0) = a$ (the limit of the tail) by construction of function.
  - A subspace A of a complete metric space M is itself complete if and only if it is a closed set in M.
  - In a finite-dimensional vector space, every metric associated to a norm is complete. (no proof).
- Closed vs. Complete

- A metric space *M* is always closed *M*. A metric space *M* may or may not complete.
- A set *A* is complete iff every Cauchy sequence has a limit in *A*.

  A set *A* is closed in *M* iff Cauchy sequence $(x_n)_{n=1}^{\infty}$ in *A* has limit in $M \Rightarrow (x_n)_{n=1}^{\infty}$ has limit in *A*.

  Ex. $M=(0,2]$, $A=(0,1]$. The point $0 \notin A$ is a limit point of *A* in $\mathbb{R}$, thus *A* is not closed in $\mathbb{R}$. However, $0 \notin M$; thus, 0 is not a limit point (in *M*) of *A*. In fact, $(0,1]$ is closed in $(0,2]$.
- A subspace *A* of a complete metric space *M* is itself complete if and only if it is a closed set in *M*.

- Def: The **completion** $\overline{M}$ of *M* is the set of equivalence classes of Cauchy sequences of points in *M*.

  We regard *M* as a subset of $\overline{M}$ by identifying the point *x* in *M* with the equivalent class of the sequence $(x,x,\ldots)$.

  We can make $\overline{M}$ into a metric space by defining the distance between the equivalent class of $(x_1,x_2,\ldots)$ and the equivalence class of $(y_1,y_2,\ldots)$ to be $\lim_{n\to\infty} d(x_n,y_n)=0$.

  This definition requires that we verify
  1) the limit exists
  2) the limit is independent of the choice of sequences from the equivalence classes, and
  3) the distance so defined satisfies the axioms for a metric.
- Def: A complete normed vector space is called a **Banach** space.
- Def: A complete inner product space is called a **Hilbert** space.
- Def: If *A* is a subset of *M*, we say $\mathcal{B}$, a collection of subsets *B* of *M*, is a **covering** if *A* if $A \subseteq \bigcup_{\mathcal{B}} B$ and an open covering if all the sets *B* are open sets in *M*.

  A subcovering means a subcollection $\mathcal{B}'$ of $\mathcal{B}$
- **Boundedness**:
  - $\exists x \in A\ \exists R > 0$ finite such that $\forall y \in A$, $d(x,y) < R$.
  - $\exists x \in A\ \exists R > 0$ finite such that $B_R(x) = A$.
    - The inf of such *R* defines the radius of the space with respect to *x*.
    - the radius is finite with respect to every point in the space (≤ 2*R*).
  - Let $D = \sup_{x,y} d(x,y)$ be the diameter of the space.

    The diameter is finite iff the radius is finite.
  - For any given *x*, let *R* be a radius with respect to *x*. Then

$R \le D \le 2R$.

- Heine-Borel property: every open covering has a finite subcovering
  - If *A* is a subspace of *M*, the the Heine-Borel property for *A* as a subspace of *M* (open meaning open in *M*) is equivalent to the Heine-Borel property for *A* as a subspace of *A*.

- **compact**
  - A is compact (A is a compact subset of a metric space)
    - (Def) if every sequence $a_1, a_2, \ldots$ of points in *A* has a limit point in *A*.

    ≡ every sequence $a_1, a_2, \ldots$ of points in *A* has a subsequence that converges to a point in *A*.

    ≡ (**Heine-Borel**) *A* has the Heine-Borel property:
    every open covering has a finite subcovering
      - If *A* is a subspace of *M*, then the Heine-Borel property for *A* as a subspace of *M* (open meaning open in *A*) is equivalent to the Heine-Borel property for *A* as a subspace of *A*.

    ≡ *A* is bounded, complete, and $\forall \frac{1}{m}$ there exists a finite subset $x_1, \ldots, x_n$ such that every point in *A* is within distance $\frac{1}{m}$ of one of them.
  - It is the same thing to say *A* is a compact subset of *M* or *A* is a compact subset of *N* if *N* is any subspace of *M* containing *A*.
  - Def: A metric space *M* is compact
    - if *M* is a compact subset of itself

    ≡ if all sequences of points in *M* have limit points in *M*.
  - A is a compact subset of *M* if and only if *A* as a subspace is a compact metric space.
  - *A* is a compact metric space ⇒
    - *A* is complete (converse not true. Ex. $\mathbb{R}$)
    - *A* has a countable dense subset
    - *A* is bounded.
    - $\forall \frac{1}{m}$ there exists a finite set of points $x_1, \ldots, x_n$ such that every point is within distance $\frac{1}{m}$ of one of them ⇒ $B_{\frac{1}{m}}(x_1), \ldots, B_{\frac{1}{m}}(x_n)$ covers the space.
  - A subspace of $\mathbb{R}^n$ is compact if and only if it is closed (complete) and bounded.
    - This is not true of general metric space.
  - *X* compact. $A \subset X \subset M$. *A* closed in *M*. ⇒ *A* is compact.

(Closed subsets of compact sets are compact.)

Proof. Sequence $\{x_n\}$ in A is sequence in X. By compactness of X, $\{x_n\}$ has limit point in X, which is also a limit point (in M) of A. A is closed; thus, the limit point is in A.

- X compact. $X \subset M$. $\Rightarrow X$ is closed in M.
  (Compact subsets of metric spaces are closed.)

  Proof. Let $x \in M$ be a limit point of X. Then, $\exists$ sequence in X $\{x_n\} \to x$. So, x is a limit point of a sequence in
  By compactness of X, x is in X.

- $C([a,b])$ with sub-norm metric $d(f,g) = \sup_x |f(x) - g(x)|$
  - complete
  - Let $f_1, f_2, \ldots$ be a sequence of continuous function converging Pointwise to a discontinuous function. Let A be the set $\{f_1, f_2, \ldots\}$. Then, A is bounded, closed (no limit point), and not compact ($f_1, f_2, \ldots$ is a sequence from A with no convergent subsequent.)
- Def: A sequence of function $\{f_k\}$ on a domain D is said to be **uniformly bounded** if $\exists M$ such that $|f_k(x)| \le M$ for all k and all x in D.
- Def: A sequence of function $\{f_k\}$ on a domain D is said to be **uniformly equicontinuous** if $\forall \frac{1}{m} \exists \frac{1}{n}$ such that $|x-y| < \frac{1}{n} \Rightarrow \forall k \, |f_k(x) - f_k(y)|$.
- Arzela-Ascoli theorem: A sequence of uniformly bounded and uniformly equicontinuous functions on a compact interval has a uniformly convergent subsequent.
- **Equivalent** metrics
  - Def: two metrics d_1 and d_2 on the same set M is equivalent
    - if $\exists c_1, c_2 > 0$ such that $\forall x, y \in M$, $d_1(x,y) \le c_2 d_2(x,y)$ and $d_2(x,y) \le c_1 d_1(x,y)$

    $\equiv$ $\exists a, b > 0$ such that $a\, d_2(x,y) \le d_1(x,y) \le b\, d_2(x,y)$.
  - If d_1 and d_2 are equivalent,
    - $x_n \to x$ in d_1-metric iff $x_n \to x$ in d_2-metric.
    - then they have the same open sets.
  - Any metrics associated with a norm on $\mathbb{R}^n$ are equivalent.

Continuous Functions on Metric Spaces

- $f : M \to N$. $A \subset M$. $f(A) \subset B \Rightarrow A \subset f^{-1}(B)$

- $f: M \to N$. $A \subset f^{-1}(B) \underset{\text{implicitly implies}}{\subset} M \Rightarrow$
- Def:
  - $f: M \to N$ means *f* is a function whose domain is *M* and whose range is *N*, where both *M* and *N* are metric spaces.
  - The image $\boxed{f(M) = \{y \in N : \exists x \in M \text{ such that } f(x) = y\}} \subset N$.
  - $f(M) = N$ iff *f* is onto.
  - $\boxed{f^{-1}(B) = \{x \in M : f(x) \in B\} = f^{-1}(B \cap f(M))} \subset M$.
  - $f^{-1}(f(M)) = M$
  - $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

    $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

    $f(A \cup B) = f(A) \cup f(B)$

    The statement $f(A \cap B) = f(A) \cap f(B)$ is false. Consider $A = \{1,2\}$, $B = \{2,3\}$, and $f(1) = f(3) = a$, $f(2) = b$.
  - $f: M \to N$
    - $f^{-1}(N) = M$
    - $f(f^{-1}(N)) = f(M) \subset N$
    - If $A \cup B = N$ then $M = f^{-1}(N) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

- **Continuous**

  Let *M* and *N* be metric spaces, $f: M \to N$ a function.

  The following three conditions are equivalent (and a function satisfying them is called continuous.)

  1) $\forall \frac{1}{m}$ and x_0 in *M*, $\exists \frac{1}{n}$ such that $d^M(x, x_0) \le \frac{1}{n} \Rightarrow d^N(f(x), f(x_0)) \le \frac{1}{m}$.

  $\equiv$ $\forall \frac{1}{m}$ and x_0 in *M*, $\exists \frac{1}{n}$ such that $x \in B^M_{\frac{1}{n}}(x_0) \Rightarrow x \in B^N_{\frac{1}{m}}(x_0)$.

  $\equiv$ $\forall \frac{1}{m}$ and x_0 in *M*, $\exists \frac{1}{n}$ such that $f\left(B^M_{\frac{1}{n}}(x_0)\right) \subset B^N_{\frac{1}{m}}(x_0)$.

  2) If $x_1, x_2, \ldots$ is any convergent sequence in *M*, then $f(x_1), f(x_2), \ldots$ is convergent in *N*.

  $\Rightarrow \lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right)$.

3) If B is any open set in N, then $f^{-1}(B)$ is open in M.

- Note: When $M \subset \mathbb{R}$ and M is open in $\mathbb{R}$,
  $f^{-1}(B)$ is open in $M \Leftrightarrow f^{-1}(B)$ is open in $\mathbb{R}$.
  B open in $N \Leftrightarrow B$ open in $\mathbb{R}$.
- In stead of N, we can use any set N' containing $f(M)$:
  It is immaterial whether we take the range N as given, or reduce it to $f(M)$, or enlarge it to some space containing N, as long as we keep the same metric on the image.
  - B open in $N' \Rightarrow B \cap f(M)$ open in $f(M)$.
  - $f^{-1}(B) = \{x \in M : f(x) \in B\} = f^{-1}(B \cap f(M))$.

- $f: M \to N$ is continuous $\Rightarrow \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$.
- Example of continuous function
  - Consider (M,d) and $(\mathbb{R}, |\cdot|)$. Let $x_0 \in M$, then $f(x) = d(x, x_0) : M \to \mathbb{R}$ is continuous.
  - $d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$. Then any function f: $(M,d) \to (N, d^N)$ is continuous.

    Set δ < 1, then $d(x,x_0) \le d \Rightarrow d(x,x_0) < 1 \Rightarrow d(x,x_0) = 0 \Rightarrow x = x_0$. So, $d^N(f(x), f(x_0)) = d^N(f(x_0), f(x_0)) = 0 \le e$
    - This includes $(M,d) \to (\mathbb{R}, |\cdot|)$, $(\mathbb{R}, d) \to (\mathbb{R}, |\cdot|)$.
  - $f(x) = \begin{cases} 0, & x = x_0 \\ 1, & x \neq x_0 \end{cases}$ is not continuous from $(\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ nor $(M, |\cdot|) \to (\mathbb{R}, d)$.
  - Coordinate projection maps: $f: (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}, |\cdot|)$. $f(x) =$ the k^{th} coponent of x.
  - Let $a = (a_1, \ldots, a_n)$ denote n-tuple of non-negative integers (each a_k can equal 0, 1, 2, …), and let $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Then, $p(x) = \sum c_a x^a$, were the sum is finite and c_a are constants, is the general **<u>polynomial</u>**s on $\mathbb{R}^n$. Let $|a| = \sum_{i=1}^{n} a_i = \|a\|_1$.

    We call x^a a **<u>monomial</u>** of order or degree $|a|$, and we call the **<u>order</u>** of the polynomial the order of the highest monomial appearing in it with non-zero coefficient.

- Let $f: D \to \mathbb{R}$ where $D \subset \mathbb{R}$. $f(x) = \begin{cases} g(x), & x \text{ is rational} \\ h(x), & \text{otherwise} \end{cases}$. $g(x)$ and $h(x)$ are continuous from $\mathbb{R} \to \mathbb{R}$. Then, $f(x)$ is continuous at $x_0 \in \mathbb{R}$ if and only if $g(x_0) = h(x_0)$.

  "$\Leftarrow$": $g(x_0) = h(x_0) = a$. Then, by the continuity of $g(x)$ and $h(x)$, given e, can find $d > 0$ such that $|x - x_0| < d$ implies both $|g(x) - a| < \frac{e}{2}$ and $|h(x) - a| < \frac{e}{2}$. Now, given x, $|f(x) - f(x_0)|$ can be one of the four possibilities: $|g(x) - g(x_0)|$, $|h(x) - h(x_0)|$, $|g(x) - h(x_0)|$, and $|h(x) - g(x_0)|$, depending on the rationality of x and x_0. Whatever the form of $|f(x) - f(x_0)|$ is, they are all $< e$ if we keep $|x - x_0| < d$.

  "$\Rightarrow$": Assume $g(x_0) > h(x_0)$. Then, let $e = \frac{g(x_0) - h(x_0)}{3} > 0$. Then, for d small enough, by the continuity of $g(x)$ and $h(x)$, $|x - x_0| < d$ implies that $g(x) - h(x) > e = \frac{g(x_0) - h(x_0)}{3}$.

- Continuous functions are closed under
  - restriction to a subspace
  - composition
  - addition (when the range is $\mathbb{R}^n$) and
  - multiplication (when the range of one is $\mathbb{R}$ and the other $\mathbb{R}^n$).
- If $f_k : (M, d) \to (\mathbb{R}, |\cdot|)$ for $k = 1, \ldots, n$ and $f(x) = (f_1(x), \ldots, f_n(x)) : (M, d) \to (\mathbb{R}^n, |\cdot|)$, the f is continuous if and only if all $f_k : M \to \mathbb{R}$ are continuous.
- Example of not continuous function
  - $f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$ is not continuous at the origin, but is continuous in x for each fixed y and continuous in y for each fixed x.
- Def. $f : (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}, |\cdot|)$ is **<u>separately continuous</u>** if $\forall k$ and every fixed value of all x_j with $j \neq k$, the function $g(x_k) = f(x_1, \ldots, x_n)$: $(\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$ is continuous.
- Continuity implies separate continuity.

- $f:(\mathbb{R}^n,|\cdot|)\to(\mathbb{R},|\cdot|)$ continuous $\Rightarrow$ $g(x_k)=f(x_1,\ldots,x_n):(\mathbb{R},|\cdot|)\to(\mathbb{R},|\cdot|)$ continuous.
- Def: $f:M\to N$ is said to be **uniformly continuous** if $\forall\frac{1}{m}\ \exists\frac{1}{n}$ such that $\forall x,y\in M$, $d(x,y)\le\frac{1}{n}\Rightarrow d(f(x),f(y))\le\frac{1}{m}$.
- Continuous function and compact set.
    - Let M be compact. Then $f:M\to N$ continuous implies it is uniformly continuous.
      So, M compact, then $f:M\to N$ uniformly continuous iff continuous.
    - If M is compact and $f:M\to\mathbb{R}$ is continuous, then $\sup_x f(x)$ and $\inf_x f(x)$ are finite and there are points in M where f attains these values.
    - The image of a compact set under a continuous function is compact.
- **Connected space**
    - M is connected
        - (Def) if there do not exist disjoint nonempty open (in M) sets A and B with $M=A\cup B$.
            - $A=B^c=M/B\neq\emptyset,M$ open and closed (clopen).
              $B=A^c=M/A\neq\emptyset,M$ open and closed (clopen).
            - The pair A and B is called a disconnection of M.

        $\equiv$ the only sets both open and closed (clopen) in M are the empty set and M.

            - If M is not connected, then the A and B from the definition of M are two sets that are both open and closed and not equal to $\emptyset,M$.
    - (being of one piece; impossibility of splitting the space up into pieces.)
    - Not a relative property for metric spaces.
    - M is disconnected if and only if there exists a continuous map from M onto $\{0,1\}$.
    - Example of connected spaces
        - $\mathbb{R}$
        - A subspace of $\mathbb{R}$ is connected if and only if it is an **interval**.
    - Example of disconnected spaces.
        - A discrete space containing two or more points
    - I is an interval iff $\forall a,b\in I$, $a<b$, $\forall c\in\mathbb{R}$, $a<c<b\Rightarrow c\in I$.
        - If c is not in I, then $I=\underbrace{\left(I\cap(-\infty,c)\right)}_{\text{open in } I,\ \text{contain } a}\cup\underbrace{\left(I\cap(c,\infty)\right)}_{\text{open in } I,\ \text{contain } b}$; not connected.

          disjoint

- **Curve** (or **arc, path**)
  - Def: a curve in M is a continuous function from an interval (in $\mathbb{R}$) to M. $\left(f : I \to M \right)$
  - Think of $f\left(I\right)$ as being traced out by $f(t)$ as t varies in I, interpreted as a time variable. Thus, the curve is a "trajectory of a moving particle" in M.
  - When M is a subspace of $\mathbb{R}^n$, the curve has the form $f\left(t\right) = \left(f_1\left(t\right), f_2\left(t\right), \ldots, f_n\left(t\right)\right)$ where $f_k(t)$ are continuous numerical functions, giving the coordinates of the trajectory at each time t.
  - The graph of a continuous function $g : I \to \mathbb{R}$ is a curve $\left(M = \left\{ \left(x, g\left(x\right)\right); x \in I \right\} \right)$ in the plane given by $f\left(t\right) = \left(t, g\left(t\right)\right)$ for t in I.
- **Arcwise (path wise) connected**
  - A space M is arcwise connected
    - (def) if there exists a curve connecting any two points.
    - (def) if $\forall x, y \in M$, there exists a curve (continuous function) $f : \left[a,b\right] \to M$ with $f\left(a\right) = x$, $f\left(b\right) = y$

    $\equiv$ if $\forall x, y \in M$, there exists a curve (continuous function) $g : \left[a',b'\right] \to M$ with $g\left(a'\right) = x$, $g\left(b'\right) = y$.

    Let $h\left(t\right) = a + \dfrac{b-a}{a'-b'}\left(t - a'\right) : \left[a',b'\right] \xrightarrow{\text{onto}} \left[a,b\right]$.

    Let $g(t) = f\left(h\left(t\right)\right)$; continuous because f and h are continuous.

    $g\left(b'\right) = f\left(h\left(b'\right)\right) = f\left(b\right) = y$.
  - being able to join any two points by a continuous curve.
  - Let X be a metric space and let $x_0, x_1, x_2 \in X$. Suppose that there is a curve connecting x_0 and x_1 and another curve connecting x_1 and x_2. Then, there is a curve connecting x_0 and x_2.

    $\exists$ continuous $f : \left[0,1\right] \to X$, $f\left(0\right) = x_0$, $f\left(1\right) = x_1$.

    $\exists$ continuous $g : \left[1,2\right] \to X$, $g\left(1\right) = x_1$, $g\left(2\right) = x_2$.

    Let $h : \left[0,2\right] \to X$, $h\left(t\right) = \begin{cases} f\left(t\right), & 0 \le t < 1 \\ x_1, & t = 1 \\ g\left(t\right), & 1 < t \le 2 \end{cases}$. We need to show that h is continuous.

    Because f and g are continuous, $\forall t \ne 1$, we know that $h\left(t\right)$ is continuous. At t = 1, because f and g are continuous, $\forall e$, we know that $\exists d_1$

$t \in (1-d_1, 1] \Rightarrow |f(t) - f(1)| < e$ and $\exists d_2 \;\; t \in [1, 1+d_2) \Rightarrow |g(t) - g(1)| < e$.

Now, choose $0 < d < d_1, d_2$. Then, for all $t \in (1-d, 1+d)$,

$t \in (1-d, 1] \subset (1-d_1, 1] \Rightarrow |h(t) - h(1)| = |f(t) - x_1| < e$,

$t = 1 \Rightarrow |h(t) - h(1)| = 0 < e$,

$t \in [1, 1+d) \subset [1, 1+d_2) \Rightarrow |h(t) - h(1)| = |g(t) - x_1| < e$,

So, $\forall e \; \exists d$ such that $t \in (1-d, 1+d) \Rightarrow |h(t) - h(1)| < e$. Therefore, $h(t)$ is also continuous at $t = 1$.

- If M is arcwise connected, and $g: M \to \mathbb{R}$ is any continuous real-valued function, then g has the intermediate value property.

- Arcwise connected implies connected.

- Let $f: M \to N$ be continuous and onto (surjective) $(f(M) = N)$.
  - If M is connected, then so is N.

N not connected $\Rightarrow N = \underbrace{\underset{\substack{\text{open in } N \\ \text{nonempty}}}{A} \cup \underset{\substack{\text{open in } N \\ \text{nonempty}}}{B}}_{\text{disjoint}}$.

$\exists a \in N \; a \in A$. By onto, $\exists x \in M \;\; f(x) = a$. Thus, $f^{-1}(A) \neq \varnothing$. Similarly, $f^{-1}(B) \neq \varnothing$. Also, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varnothing) = \varnothing$. By continuity of f, $f^{-1}(A)$ and $f^{-1}(B)$ are open. Thus,

$M = f^{-1}(N) = f^{-1}(A \cup B) = \underbrace{\underset{\substack{\text{open in } M \\ \text{nonempty}}}{f^{-1}(A)} \cup \underset{\substack{\text{open in } M \\ \text{nonempty}}}{f^{-1}(B)}}_{\text{disjoint}}$, not connected.

  - If M is arcwise connected, then so is N.

- Fixed points: $f(x) = x$.
- **Contractive mapping**
  - Consider a function whose domain and range are of the same metric space. which we assume is complete.
  - Def: Let (M,d) be a metric space. $f: M \to M$ is a **contractive mapping** if $\exists r < 1$ such that $\forall x, y \in M \;\; d(f(x), f(y)) \le r d(x, y)$.
    - $\Rightarrow$ continuity (Lipschitz condition with constant < 1)

      $f\left(\lim_{n\to\infty} f^n(x)\right) = \lim_{n\to\infty} f^{n+1}(x)$
    - Not work when having $d(f(x), f(y)) \le d(x, y)$ or even $d(f(x), f(y)) < d(x, y)$.

- The map $f: M \to M$ is a contraction.
- $\Rightarrow$ shrinking map.

- Def: $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$

- **Contractive mapping principle**: Let M be a complete metric space and $f: M \to M$ a contractive mapping. Then,
  - there exists a unique fixed point x_0, and $x_0 = \lim_{n\to\infty} f^n(x)$ $\forall x \in M$, with $d(x_0, f^n(x)) \le cr^n$ for a constant c depending on x.

  - Compact M with contractive mapping will work also because compact $\Rightarrow$ complete.
  - $d(f^{n+1}(x), f^n(x)) \le rd(f^n(x), f^{n-1}(x)) \le \cdots \le r^n d(f(x), x)$
  - $m > n$: $d(f^m(x), f^n(x)) \le \left(\sum_{k=n}^{m-1} r^k\right) d(f(x), x)$

    $$\le \left(\sum_{k=n}^{\infty} r^k\right) d(f(x), x) = \frac{r^n}{1-r} d(f(x), x)$$
  - $d(f^n(x), x_0) = d\left(f^n(x), \lim_{m\to\infty} f^m(x)\right) = \lim_{m\to\infty} d(f^n(x), f^m(x))$

    $$\le \frac{r^n}{1-r} d(f(x), x)$$

- $f: [a,b] \to [a,b]$ is continuous on $[a,b]$, differentiable on (a,b), and has $|f'(x)| \le a < 1$ for all $a < x < b$. Then, f has a unique fixed point.

  By the mean value theorem, $\forall x, y$ $\exists x_0$ $\frac{f(x) - f(y)}{x - y} = f'(x_0)$. So, $\frac{|f(x) - f(y)|}{|x - y|} = |f'(x_0)| \le a < 1$.

- Let $f: X \xrightarrow[\text{onto}]{1:1} X$, $g = f^{-1}: X \to X$. Then x_0 is a unique fixed point of $g \Leftrightarrow x_0$ is a unique fixed point of f.

  Proof. "$\Rightarrow$" x_0 is a fixed point of $g \Rightarrow$ $g(x_0) = x_0$ $\Rightarrow$ $x_0 = f(x_0)$ $\Rightarrow x_0$ is a fixed point of f. Let x_1 be any fixed points of f, then $f(x_1) = x_1$, which implies $g(x_1) = x_1$. By the uniqueness of the fixed point of g, we have $x_1 = x_0$.

- Let (X,d) be a complete metric space and $f: X \to X$ be surjective. Assume that there exists $c > 1$ such that $d(f(x), f(y)) \ge cd(x,y)$ $\forall x, y \in X$. Then, 1) f is injective and 2) has a unique fix point.

Proof 1): Consider any $x \neq y$. So, $d(x,y) > 0$. $\Rightarrow d(f(x), f(y)) \geq cd(x,y) > 0$. $\Rightarrow f(x) \neq f(y)$.

Proof 2) Define $g = f^{-1} : X \to X$. g is a contractive mapping. Consider any x, y. Let $g(x) = a$ and $g(y) = b$. Then, $d(g(x), g(y)) = d(a,b) \leq \frac{1}{c} d(f(a), f(b))$ $= \frac{1}{c} d(x,y)$. Note that $0 < \frac{1}{c} < 1$. So, there exists a unique fixed point x_0; $g(x_0) = x_0$. Hence, x_0 is a unique fixed point of f.

- Def: Let (M,d) be a metric space. A map $f : M \to M$ is a **shrinking map** if $\forall x, y \in M$ if $x \neq y$, $d(f(x), f(y)) < d(x,y)$
  - The function $g(x) = d(f(x), x) : (M, d) \to (\mathbb{R}, |\cdot|)$ is continuous.

    Using the quadrilateral inequality, $|g(x) - g(y)| = |d(f(x), x) - d(f(y), y)|$ is $\leq d(x,y) + d(f(x), f(y))$. From def, this is $\leq 2d(x,y)$. (Lipschitz).
- If f is a shrinking map and M is compact, then f has a unique fixed point.

  Because $g(x) = d(f(x), x)$ is continuous and M is compact, $\exists x_0 \in M$ such that $g(x_0) = \inf_{x \in M} g(x)$. This x_0 is the fixed point $(d(f(x_0), x_0) = 0)$. If not, then $g(f(x_0)) = d(f(f(x_0)), f(x_0)) < d(f(x_0), x_0) = g(x_0)$, contradiction because minimum of g is attained at x_0, $g(f(x_0))$ can't be lower than $g(x_0)$.

  Uniqueness: If $x_1 \neq x_2$ are both fixed points, then $d(x_1, x_2) = d(f(x_1), f(x_2)) < d(x_1, x_2)$.
- Brouwer fixed point theorem: there is always a fixed point (not necessarily unique) if M is a closed ball in $\mathbb{R}^n$.
  - There does not have to be a fixed point if M is an open ball.