- **Cantor set C_0**
  - The Cantor set is
    - the subset of [0,1] of all numbers expressible in base 3 with digits 0 and 2. ⇒ $(0.a_1a_2\ldots)_3$, $a_k \in \{0,2\}$.
    - what's left over after removal of a sequence of open subintervals of [0, 1]. The algorithm is as follows: 1) Divide the remaining intervals each into three equal parts. 2) Remove the open middle interval. 3) Repeat 1).
  - countable intersection of finite unions of closed intervals.
  - closed (intersection of closed sets), non-empty, perfect
  - Doesn't contain any interval.
  - Has no isolated point.
  - Has the same cardinality as $\mathbb{R}$. $(0.a_1a_2\ldots)_3 \xrightarrow[onto]{1:1} (0.b_1b_2\ldots)_2$, $b_k = \frac{a_k}{2} \in \{0,1\}$.

The Concept of Measure

Intervals

- I denotes an interval (a,b) or [a,b], or (a,b] or [a,b), then length of I = $|I|$ = $b - a$.
  - If a = -∞ or b = +∞, then $|I|$ = +∞.
- $|I|$ is a non-negative extended real number.
- Additivity: $I = I_1 \cup I_2$, $I_1 \cap I_2 = \varnothing$, then $|I| = |I_1| \cup |I_2|$.
- Finite additivity of length: If interval $I = \bigcup_{i=1}^{n} I_i$ is a <u>disjoint union</u> of intervals, then

  $|I| = \sum_{i=1}^{n} |I_i|$.

  Proof. Induction. Observe that it is possible to remove one of the intervals I_k (say the one containing an endpoint or a neighborhood of an endpoint if I is open) so that the union of the remaining intervals is still an interval.
- Subadditivity: interval $I \subset \bigcup_{i=1}^{n} I_i \Rightarrow |I| \le \sum_{i=1}^{n} |I_i|$.

  Proof. Shirk I_j to I_j' so that I is the disjoint union $I = \bigcup_{i=1}^{n} I_i'$.
  - Monotonicity: $I \subset J \Rightarrow |I| \le |J|$.

- Countable additivity / s-additivity: If interval $I = \bigcup_{i=1}^{\infty} I_i$ is a disjoint union of intervals, then $|I| = \sum_{i=1}^{\infty} |I_i|$.
  - If one side is +∞, then so is the other.
  - $\sum_{i=1}^{\infty} |I_i|$ can be +∞ either because one of the summands is +∞ or because the series diverges.

    1) Proof for finite $|I|$. "≥"Consider $\bigcup_{i=1}^{n} I_i \subset I$. Rearrange $(I_i) = (I'_k)$ so that I'_{k-1} lies to the left of I'_k. J_i's fill the gaps. Disjoint I'_k's J_i's. $I = \sum_{j=1}^{n} |I_j| + \sum_{j=1}^{n} |J_k|$.

    $I \geq \sum_{j=1}^{n} |I_j|$. Let $n \to \infty$. (limit exists because bounded and monotone increasing) "≤"

    $\forall e > 0$, compact $I' = \left[a + \frac{e}{2}, b - \frac{e}{2}\right] \subset I$. $|I'| = I - e$. open

    $I'_j = \left(a - \frac{e}{22^j}, b + \frac{e}{22^j}\right) \supset I_j$. $|I'_j| = |I_j| + \frac{e}{2^j}$. $I' \subset I = \bigcup_{i=1}^{\infty} I_i \subset \bigcup_{i=1}^{\infty} I'_i$ open cover.

    Heine-Borel theorem $\Rightarrow \exists N$ $I' \subset \bigcup_{i=1}^{N} I'_i \Rightarrow |I'| \leq \sum_{i=1}^{N} |I'_i|$. $I - e =$ $|I'| \leq \sum_{i=1}^{N} |I'_i|$

    $= \sum_{i=1}^{N} \left(|I_j| + \frac{e}{2^j}\right) \leq \sum_{i=1}^{\infty} \left(|I_j| + \frac{e}{2^j}\right) = \sum_{i=1}^{N} |I_j| + e$.

    2) Proof for $|I| = +\infty$. $\underbrace{I \cap [N, N]}_{\text{result is finite interval}} = \underbrace{\bigcup_{i=1}^{\infty} (I_i \cap [N, N])}_{\text{disjoint}}$. By case 1),

    $|I \cap [N, N]| = \bigcup_{i=1}^{\infty} |I_i \cap [N, N]| \leq \bigcup_{i=1}^{\infty} |I_i|$. $\lim_{N \to \infty} |I \cap [N, N]| = |I| = +\infty$.

- $I \subset \bigcup_{j=1}^{\infty} I_j \Rightarrow |I| \leq \sum_{j=1}^{\infty} |I_j|$

  Proof. Construct disjoint interval I'_j by reducing I_j. $\bigcup_{j=1}^{\infty} I_j = \underbrace{\bigcup_{j=1}^{\infty} I'_j}_{\text{disjoint}}$. $I = \bigcup_{j=1}^{\infty} (I'_j \cap I)$.

  $|I| = \sum_{j=1}^{\infty} |I'_j \cap I|$. $|I'_j \cap I| \leq |I'_j| \leq |I_j|$.

- $|I| = \inf \left\{ \sum_{j=1}^{\infty} |I_j| : I \subset \bigcup_{j=1}^{\infty} I_j \right\}$

Proof. Let $C=\left\{\sum_{j=1}^{\infty}\left|I_j\right| : I\subset\bigcup_{j=1}^{\infty}I_j\right\}$. "≥" $|I|\in C$ because $I\subset I\cup\varnothing\cup\varnothing\cup\cdots$. "≤" $\forall\bigcup_{j=1}^{\infty}I_j$, $I\subset\bigcup_{j=1}^{\infty}I_j\Rightarrow|I|\le\sum_{j=1}^{\infty}\left|I_j\right|$. So, $|I|$ is a lower bound of C.

s-field

- Let X be a set (universal), and $\mathcal{F}$ is a family of subsets of X. $\mathcal{F}$ is called a **field** (or algebra of sets) provided that
  1) $\varnothing\in\mathcal{F}$
  2) $A\in\mathcal{F}\Rightarrow {}^{c}A\in\mathcal{F}$.
  3) $A,B\in\mathcal{F}\Rightarrow A\cup B\in\mathcal{F}$.

  $\Rightarrow$ $A,B\in\mathcal{F}\Rightarrow A\cap B\in\mathcal{F}$

  Proof. $A\cap B={}^{c}\left({}^{c}A\cup{}^{c}B\right)$

  $\Rightarrow$ $A,B\in\mathcal{F}\Rightarrow A\setminus B\in\mathcal{F}$

  Proof. $A\setminus B=A\cap{}^{c}B$.

  - A field forms a Boolean algebra under the operation $A+B=A\Delta B=\left(A\setminus B\right)\cup\left(B\setminus A\right)$ and $A\cdot B=A\cap B$.

  Ex. Set of all subsets of X.

  Ex. X = any fixed interval of $\mathbb{R}$. $\mathcal{F}=\{\text{finite unions of intervals contained in } X\}$.

- A field $\mathcal{F}$ (on X) is called a **s-field** provided that if A_1, A_2, … is a sequence of sets in $\mathcal{F}$, then $\bigcup_{j=1}^{\infty}A_j\in\mathcal{F}$.

  $\Rightarrow$ $\bigcap_{j=1}^{\infty}A_j\in\mathcal{F}$.

  Proof. $\bigcap_{j=1}^{\infty}A_j={}^{c}\left(\bigcup_{j=1}^{\infty}{}^{c}A_j\right)$.

  Ex. Set of all subsets of X.

- Let $\mathcal{F}$ be a field. $\mathcal{F}_s$ = the **s-field generated by $\mathcal{F}$** is the intersection of all s-field containing $\mathcal{F}=\bigcap_{\substack{g\text{ is a } s\text{-field}\\ \mathcal{F}\subset g}} g$.

  - Is a s-field

    Proof. Let A, B, $A_1,A_2,\ldots\in\mathcal{F}_s$, then A, B, $A_1,A_2,\ldots$ every g. All g is s-field; thus, 1) $\varnothing$ in every g. 2) ${}^{c}A$ also in every g. 3) $A\cup B$ also in every g. 4) $\bigcup_{j=1}^{\infty}A_j$ also in every g.

- Is the smallest s-field containing $\mathcal{F}$.
  If $\mathcal{H}$ is a s-field and $\mathcal{F} \subset \mathcal{H}$, then $\mathcal{F}_s \subset \mathcal{H}$.

  Proof. $\mathcal{H}$ is one of the g's.
- Let interval $X \subset \mathbb{R}$. $\mathcal{B}_X$ = s-field generated by the field of finite unions of subintervals of X.
  - Call $\mathcal{B}_X$ the <u>**s-field of Borel subsets**</u> of X or s-field of Borel sets on X.
  - Call set $B \in \mathcal{B}_X$ <u>**Borel set**</u>.
  - Is the smallest s-field containing finite unions of subintervals of X.
- G_d set is a countable intersection of open sets
  - Ex. open sets, intervals, countable intersection of countable union of open sets.
  - $\mathbb{Q}$ is not G_d.
  - G_d is not a field; thus, not a s-field.

Measure

- Let $\mathcal{F}$ be a s-field and $m : \mathcal{F} \to [0,\infty]$. m is called a <u>**measure**</u> provided that

  1) $m(\varnothing) = 0$

  2) s**-additivity**: if $A = \bigcup_{j=1}^{\infty} A_j$ with A_j disjoint, then $|A| = \sum_{j=1}^{\infty} |A_j|$.

- **Finite additivity**: if $A = \bigcup_{j=1}^{n} A_j$ with A_j disjoint, then $|A| = \sum_{j=1}^{n} |A_j|$.

  Proof. From s-additivity, let $A_j = \varnothing \quad \forall j \geq n+1$.
- Members of $\mathcal{F}$ is called **measurable sets**
- **Monotonicity**: $A, B \in \mathcal{F}$, $A \subset B \Rightarrow |A| \leq |B|$.

  Proof. $|B| = |A| + \underbrace{|B \setminus A|}_{\geq 0}$.
- **Continuity from below**: If $A_1 \subset A_2 \subset A_3 \subset \cdots$ is an increasing sequence of measurable sets and, then $\left|\bigcup_{j=1}^{\infty} A_j\right| = \lim_{j\to\infty} |A_j|$.

  Proof. Let $A = \bigcup_{j=1}^{\infty} A_j$, $B_1 = A_1$, and $B_k = A_k \setminus A_{k-1}$. Then B_k's are disjoint. $A_n = \bigcup_{i=1}^{n} B_i \Rightarrow$
  $|A_n| = \sum_{i=1}^{n} B_i$. $A = \bigcup_{i=1}^{\infty} B_i \Rightarrow |A| = \sum_{i=1}^{\infty} B_i$.

- **Conditional continuity from above**. If $B_1 \supset B_2 \supset B_3 \supset \cdots$ is a decreasing sequence of measurable sets, and $|B_i|$ are finite $\forall i$, then $\left|\bigcap_{i=1}^{\infty} B_i\right| = \lim_{j\to\infty}|B_j|$.
  - Let $B = \bigcap_{i=1}^{\infty} B_i$. Then $|B|$ is finite because $B \subset B_i$.

  Ex. $B_n = (n, \infty)$. Then $\bigcap_{i=1}^{\infty} B_i = \varnothing$. $\lim_{n\to\infty}|B_n| = +\infty$.

  Proof. Let $A_k = B_k \setminus B_{k+1}$. Then 1) $B_1 = B \cup \bigcup_{j=1}^{\infty} A_j$, disjoint union $\Rightarrow |B_1| = |B| + \sum_{j=1}^{\infty} A_j$. $|B|$ is finite $(|B_1| - |B|$ is not $\infty - \infty)$; so, $|B_1| - |B| = \sum_{j=1}^{\infty}|A_j|$. 2) $B_1 = B_n \cup \bigcup_{j=1}^{n-1}|A_j|$, disjoint union $\Rightarrow |B_1| = |B_n| + \sum_{j=1}^{n-1}|A_j|$. $|B_n|$ finite; so, $|B_1| - |B_n| = \sum_{j=1}^{n-1}|A_j|$. From 1) and 2),
  $\lim_{n\to\infty}(|B_1| - |B_n|) \overset{1)}{=} \sum_{j=1}^{\infty}|A_j| \overset{2)}{=} |B_1| - |B|$.

- **Subadditivity**: If $A_1, \ldots, A_n$ are measurable sets, not necessarily disjoint, then $\left|\bigcup_{i=1}^{n} A_i\right| \le \sum_{i=1}^{n}|A_i|$.

  Proof. Let $B_1 = A_1$. $B_i = A_i \setminus \bigcup_{j=1}^{n-1} A_j$. Then, B_i's are disjoint, $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$, and $B_i \subset A_i$.

  Thus, $\left|\bigcup_{i=1}^{n} A_i\right| = \left|\bigcup_{i=1}^{n} B_i\right| = \sum_{i=1}^{n}|B_i| \le \sum_{i=1}^{n}|A_i|$.

- s-**subadditivity**: If $A_1, A_2, \ldots$ is a sequence of measurable sets, not necessarily disjoint, then $\left|\bigcup_{i=1}^{\infty} A_i\right| \le \sum_{i=1}^{\infty}|A_i|$.

  Proof. Let $B_1 = A_1$. $B_i = A_i \setminus \bigcup_{j=1}^{n-1} A_j$. Then, B_i's are disjoint, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$, and $B_i \subset A_i$.
  $\left|\bigcup_{i=1}^{\infty} A_i\right| = \left|\bigcup_{i=1}^{\infty} B_i\right| = \sum_{i=1}^{\infty}|B_i| \le \sum_{i=1}^{\infty}|A_i|$.

- If $B \subset \bigcup_{i=1}^{n} A_i$, then $|B| \le \sum_{i=1}^{n}|A_i|$.

  Proof. By monotonicity, and subadditivity.

- If $B \subset \bigcup_{i=1}^{\infty} A_i$, then $|B| \le \sum_{i=1}^{\infty}|A_i|$.

  Proof. By monotonicity, and s-subadditivity.

- If $B_1, B_2, \ldots$ is a sequence of measurable sets, and $\forall i \ |B_i| = 0$. Then, $\left|\bigcup_{i=1}^{\infty} B_i\right| = 0$.

  Proof. By s-subadditivity.

- **Outer regularity**: Let B be a Borel set. Then, $|B| = \inf\{|A| : B \subset A,\ A \text{ open}\}$.
- **Inner regularity**: Let B be a Borel set. Then, $|B| = \sup\{|F| : F \subset B,\ F \text{ closed}\}$.

Lebesgue measure

- Lebesgue measure on $\mathbb{R}$: $m : \mathcal{B}_{\mathbb{R}} \to [0, +\infty]$. $B \in \mathcal{B}_{\mathbb{R}}$.

  $m(B) = \inf\left\{\sum_{j=1}^{\infty} |I_j|;\ B \subset \bigcup_{j=1}^{\infty} I_j,\ I_j \text{ intervals}\right\}$.

  - Is a measure.

- $m(B) = \inf\left\{\sum_{j=1}^{\infty} |I_j|;\ B \subset \bigcup_{j=1}^{\infty} I_j,\ I_j \text{ disjoint intervals}\right\}$
- Outer regularity of the Lebesgue measure: Let $B \in \mathcal{B}_{\mathbb{R}}$, then $m(B) = \inf\{|A| : B \subset A,\ A \text{ open}\}$.
- Let $B \in \mathcal{B}_{\mathbb{R}}$, assume $m(B) < \infty$. Then, $\exists A \subset G_d$ such that $A \subset B$ and $m(A \setminus B) = 0$ $(m(A) = m(B))$.

Measurable function

- Let $(X, \mathcal{F}, m)$ be a measure space.
  - $\mathcal{F}$ is a s field on X.
  - Note that $\varnothing \in \mathcal{F} \Rightarrow X \in \mathcal{F}$.
- Let $\mathcal{M} = \{S \subset \mathbb{R} : f^{-1}(S) \in \mathcal{F}\}$, then $\mathcal{M}$ is a s field on $\mathbb{R}$.

  Proof. 1) $\mathcal{F}$ is a field; hence, $\varnothing \in \mathcal{F}$. $\varnothing \subset \mathbb{R}$ and $f^{-1}(\varnothing) = \varnothing \in \mathcal{F}$; hence, $\varnothing \in \mathcal{M}$. 2) Let $S \in \mathcal{F}$. Then $f^{-1}(S) \in \mathcal{F}$. $f^{-1}(\mathbb{R} \setminus S) = \underset{\in \mathcal{F}}{X} \setminus \underset{\in \mathcal{F}}{f^{-1}(S)}$. $\mathcal{F}$ is a field; hence $f^{-1}(\mathbb{R} \setminus S) \in \mathcal{F}$. Thus, $\mathbb{R} \setminus S \in \mathcal{M}$. 3) Let $S_1, S_2, \ldots \in \mathcal{M}$, $f^{-1}\left(\bigcup_{i=1}^{\infty} S_i\right) = \bigcup_{i=1}^{\infty} \underbrace{f^{-1}(S_i)}_{\in \mathcal{F}}$. $\mathcal{F}$ is a s field; hence, $f^{-1}\left(\bigcup_{i=1}^{\infty} S_i\right) \in \mathcal{F}$, and $\bigcup_{i=1}^{\infty} S_i \in \mathcal{M}$.

- Let $f : X \to \mathbb{R}$, then f is measurable

  ≡ Def: $\forall B \in \mathcal{B}_{\mathbb{R}}$ $f^{-1}(B) \in \mathcal{F}$.

  ≡ 1) $\forall I \subset \mathbb{R}$ interval $f^{-1}(I) \in \mathcal{F}$

≡ 2) $\forall I \subset \mathbb{R}$ of the form (a,∞) $f^{-1}(I) \in \mathcal{F}$

≡ 3) $\forall a \in \mathbb{R}$ $\{x \in X \,;\, f(x) > a\} \in \mathcal{F}$

Proof 1) "⇒" Trivial because $\forall I \subset \mathbb{R}$ $I \in \mathcal{B}_{\mathbb{R}}$. "⇐" We have the set $\mathcal{I}$ = $\{\text{finite unions of intervals in } \mathbb{R}\} \subset \mathcal{M}$. Note that $\mathcal{B}_{\mathbb{R}}$ is the smallest s field on $\mathbb{R}$ that contains $\mathcal{I}$. $\mathcal{M}$ is a s field on $\mathbb{R}$; hence $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}$.

Proof 2) "⇒" trivial because (a,∞) is an interval. "⇐"We have the set $\mathcal{I}'$ = $\{\text{finite unions of intervals of the form } (a,\infty) \text{ in } \mathbb{R}\} \subset \mathcal{M}$. $\mathcal{M}$ is a s field on $\mathbb{R}$; hence $\mathcal{M}$ contains $\mathbb{R} \setminus (a,\infty) = (-\infty, a]$, $(a,\infty) \setminus (b,\infty) = (a,b]$, $\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, \infty \right] = [a,\infty)$ etc. $\forall a, b \in \mathbb{R}$. (We can get any kind of intervals from elementary set operations of sets of the form (a,∞).)

- $f, f_n : X \to \mathbb{R}$, then
  - $f^{-1}(\mathbb{R} \setminus S) = X \setminus f^{-1}(S)$.

    Proof. $x \in f^{-1}(\mathbb{R} \setminus S) \Leftrightarrow f(x) \in \mathbb{R} \setminus S \Leftrightarrow f(x) \notin S \Leftrightarrow x \notin f^{-1}(S)$ $\Leftrightarrow x \in X \setminus f^{-1}(S)$.

  - $f^{-1}\left(\bigcup_{i=1}^{\infty} S_i \right) = \bigcup_{i=1}^{\infty} f^{-1}(S_i)$

    Proof. 1) $x \in f^{-1}\left(\bigcup_{i=1}^{\infty} S_i \right) \Rightarrow f(x) \in \bigcup_{i=1}^{\infty} S_i \Rightarrow \exists i_0 \; f(x) \in S_{i_0} \Rightarrow x \in f^{-1}\left(S_{i_0}\right) \Rightarrow$ $x \in \bigcup_{i=1}^{\infty} f^{-1}(S_i)$. 2) $x \in \bigcup_{i=1}^{\infty} f^{-1}(S_i) \Rightarrow \exists i_0 \; x \in f^{-1}\left(S_{i_0}\right) \Rightarrow f(x) \in S_{i_0} \Rightarrow f(x) \in \bigcup_{i=1}^{\infty} S_i$ $\Rightarrow x \in f^{-1}\left(\bigcup_{i=1}^{\infty} S_i \right)$.

  - $(g \circ f)^{-1}(A) = f^{-1}\left(g^{-1}(A)\right)$.

    Proof. $x \in (g \circ f)^{-1}(A) \Leftrightarrow g(f(x)) \in A \Leftrightarrow f(x) \in g^{-1}(A) \Leftrightarrow x \in f^{-1}\left(g^{-1}(A)\right)$.

  - $\left\{x \in X : \sup_n f_n > a\right\} = \bigcup_{k=1}^{\infty} \{x \in X : f_k(x) > a\}$

    Proof. "⊃" $x \in \bigcup_{k=1}^{\infty} \{x \in X : f_k(x) > a\} \Rightarrow \exists k_0 \; f_{k_0}(x) > a$. But $\sup_n f_n(x) \geq f_{k_0}(x)$. Hence, $\sup_n f_n(x) > a$, and therefore, $x \in \left\{x \in X : \sup_n f_n > a\right\}$. "⊂" Let

$x \in \left\{ x \in X : \sup_n f_n > a \right\}$. Assume $\forall k \in \mathbb{N}$, $f_{k_0}(x) \le a$. Then $\sup_n f_n(x) \le a \Rightarrow$ contradiction.

- Let $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$ be a measure space, and $g : \mathbb{R} \to \mathbb{R}$ continuous. Then g is measurable.

  Proof. We know that the inverse image of open set is open. $\forall a \in \mathbb{R}$ (a, ∞) is open; hence, $g^{-1}((a,\infty))$ is open. $g^{-1}((a,\infty))$ is in $\mathcal{B}_{\mathbb{R}}$ because, by structure theorem, $g^{-1}((a,\infty))$ is a (disjoint) union of at most countable number of open intervals.

- Let $(X, \mathcal{F}, m)$ be a measure space, then $f(x) \equiv c$, $c \in \mathbb{R}$ is measurable.

  Proof. $F_a = \{x \in X : f(x) > a\} = \begin{cases} \varnothing, & a \ge c \\ X, & a < c \end{cases}$. Both $\varnothing, X \in \mathcal{F}$.

- Let $(X, \mathcal{F})$ be a measurable space and $f : X \to \mathbb{R}$ measurable. Also, let $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be a measure space, and $g : \mathbb{R} \to \mathbb{R}$ measurable, then $h(x) = g \circ f(x) = g(f(x)) : X \to \mathbb{R}$ is measurable.

  Proof. $H_a = h^{-1}((a,\infty)) = f^{-1}(g^{-1}((a,\infty)))$. Because *g* is measurable, $g^{-1}((a,\infty)) \in \mathcal{B}_{\mathbb{R}}$. Also, because *f* is measurable, $\forall B \in \mathcal{B}_{\mathbb{R}}$ $f^{-1}(B) \in \mathcal{F}$.

- Let $(X, \mathcal{F})$ be a measurable space, and $f, g : X \to \mathbb{R}$ be measurable functions. Then

  I) $h(x) = f(x) + g(x) : X \to \mathbb{R}$ is measurable.

  Proof. $H_a = \bigcup_{b \in \mathbb{Q}} F_b \cap G_{a-b}$. $\forall a, b \in \mathbb{R}$ $F_b, G_{a-b} \in \mathcal{F}$. $\mathcal{F}$ is a s field; hence, $H_a = \bigcup_{b \in \mathbb{Q}} F_b \cap G_{a-b} \in \mathcal{F}$.

  II) $f^2, |f|, cf$ for $c \in \mathbb{R}$ are measurable.

  Proof. x^2, $|x|$, cx are continuous function.

  III) If $\forall x \in X$ $f(x) \ne 0$, then $h(x) = \dfrac{1}{f(x)}$ is measurable.

  Proof. For *a* > 0, we have

$$H_a = \{x \in X : h(x) > a\} = \left\{ x \in X : \frac{1}{f(x)} > a \right\}$$

$$= \left\{ x \in X : f(x) > 0, f(x) < \frac{1}{a} \right\} \cup \left\{ x \in X : f(x) < 0, f(x) > \frac{1}{a} \right\}$$

$$= \left(f^{-1}((0,\infty)) \cap f^{-1}\left(\left(-\infty, \frac{1}{a}\right)\right)\right) \cup \left(f^{-1}((-\infty,0)) \cap f^{-1}\left(\left(\frac{1}{a}, \infty\right)\right)\right)$$

  Similarly, for *a* < 0, we have

$$H_a = \left\{ x \in X : f(x) > 0, f(x) > \frac{1}{a} \right\} \cup \left\{ x \in X : f(x) < 0, f(x) < \frac{1}{a} \right\}$$
$$= \left(f^{-1}((0,\infty)) \cap f^{-1}\left(\left(\frac{1}{a}, \infty\right)\right)\right) \cup \left(f^{-1}((-\infty,0)) \cap f^{-1}\left(\left(-\infty, \frac{1}{a}\right)\right)\right)$$

For a = 0, we have $H_a = \left\{ x \in X : \frac{1}{f(x)} > 0 \right\} = \{ x \in X : f(x) > 0 \} = f^{-1}((0,\infty))$.

Because for any interval I, $f^{-1}(I) \in \mathcal{F}$ and $\mathcal{F}$ is a s field, we have $H_a \in \mathcal{F}$ since it is a union and/or intersection of sets in $\mathcal{F}$.

IV) If $a, b \in \mathbb{R}$, then $af + bg : X \to \mathbb{R}$ is measurable.

Proof. af, bg are measurable. Thus, $af + bg$ is measurable.

V) $f - g : X \to \mathbb{R}$ is measurable.

Proof. From (IV), let a = 1, b = -1.

VI) $f \cdot g : X \to \mathbb{R}$ is measurable

Proof. $f \cdot g = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$. From (I) and (V) we have $f+g$ and $f-g$ measurable. By (II), we have $(f+g)^2$ and $(f-g)^2$ measurable. By V) we have $(f+g)^2 - (f-g)^2$. By (II), we finally have $\frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$ measurable.

VII) If $\forall x \in X \; g(x) \neq 0$, then $f/g : X \to \mathbb{R}$ is measurable.

Proof. Because $\forall x \in X \; g(x) \neq 0$, from (II), $\frac{1}{g(x)}$ is measurable. Because $f(x)$ and $\frac{1}{g(x)}$ are measurable, form (VI), we conclude that $f(x)\frac{1}{g(x)}$ is measurable.

VIII) $\max(f,g) : X \to \mathbb{R}$ is measurable

Proof. First, note that $\max(f,g) = \frac{f+g}{2} + \left|\frac{f-g}{2}\right|$. Because measurability is preserved under addition, subtraction, $|\cdot|$, and scalar scaling, we conclude that $\frac{f+g}{2} + \left|\frac{f-g}{2}\right|$ is measurable.

- Let $h(x) = f(x) + g(x) : X \to \mathbb{R}$, $F_a = \{ x \in X : f(x) > a \}$, $G_a = \{ x \in X : g(x) > a \}$, $H_a = \{ x \in X : h(x) > a \} = \{ x \in X : f(x) + g(x) > a \}$, then $H_a = \bigcup_{b \in \mathbb{Q}} F_b \cap G_{a-b}$.

  Proof. We want to show that $\bigcup_{c \in \mathbb{R}} F_c \cap G_{a-c} = \bigcup_{b \in \mathbb{Q}} F_b \cap G_{a-b}$. "$\supset$" $x \in \bigcup_{b \in \mathbb{Q}} F_b \cap G_{a-b} \Rightarrow$ $\exists b_0 \in \mathbb{Q} \; f(x) > b_0$ and $g(x) > a - b_0 \Rightarrow h(x) = f(x) + g(x) > b_0 + a - b_0 = a \Rightarrow x \in H_a$.

"$\subset$" Let $x \in H_a$. Note that because $\mathbb{Q}$ is dense in $\mathbb{R}$ and $f(x) \in \mathbb{R}$, $\exists$ sequence (b_n) monotonically increasing, $\lim_{n\to\infty} b_n = f(x)$. Because $b_n < f(x)$, $x \in F_{b_n}$. Then $\exists n_0$ such that $x \in G_{a-b_n}$.

Otherwise, we have $\forall n \;\; g(x) \le a - b_n$. Take $n \to \infty$ and we have $g(x) \le a - f(x)$ which implies $h(x) = g(x) + f(x) \le a$ which contradict the assumption that $x \in H_a$.

Hence, $x \in F_{b_{n_0}} \cap G_{a-b_{n_0}} \subset \bigcup_{b\in\mathbb{Q}} F_b \cap G_{a-b}$.

- Let $(X, \mathcal{F})$ be a measurable space and $\forall n \in \mathbb{N} \;\; f_n : X \to \mathbb{R}$ measurable. Then
  - $\sup_n f_n$ is measurable

    Proof. $\left\{x \in X : \sup_n f_n > a\right\} = \bigcup_{k=1}^{\infty} \underbrace{\{x \in X : f_k(x) > a\}}_{\in \mathcal{F}}$
  - $\inf_n f_n$ is measurable
  - $\limsup f_n$ is measurable

    Proof. $\limsup f_n = \inf_n \left(\sup_{k\ge n} f_k(x) \right)$. $\sup_{k\ge n} f_k(x)$'s are measurable.
  - $\liminf f_n$ is measurable
  - If f_n converges pointwise to some function, then $\lim_{n\to\infty} f_n$ is measurable.

    Proof. $\forall x \;\; \lim_{n\to\infty} f_n(x) = \limsup f_n(x) = \liminf f_n(x)$.
- Let $(X, \mathcal{F})$ be a measurable space, $E \subset X$. Then $c_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$ is the **characteristic function** of E.
  - c_E is measurable iff E is measurable $(E \in \mathcal{F})$.

    Proof. "$\Rightarrow$" $\{1\} = [1,1] \in \mathcal{B}_{\mathbb{R}}$. c_E is measurable. Thus, $c^{-1}(\{1\}) \in \mathcal{F}$. $c^{-1}(\{1\}) = E$.

    "$\Leftarrow$" $c^{-1}((a,\infty)) = \begin{cases} \varnothing, & a \ge 1 \\ X, & a < 0 \\ E, & 0 \le a < 1 \end{cases}$. And we know that $\varnothing, X, E \in \mathcal{F}$.
- Let $(X, \mathcal{F})$ be a measurable space. $f : X \to \mathbb{R}$ is a **simple function**

  $\equiv$ (Def) $f = \sum_{i=1}^{N} a_i c_{E_i}$ where $E_i \subset X$, $\in \mathcal{F}$, and disjoint, $a_i \ne 0$.
  - Class of simple functions = Class of measurable function that take finitely many values

Proof. "⊂" $f(x)=\begin{cases} a_i, & \exists i\in\{1,\dots,N\}\ x\in E_i \\ 0, & x\notin \bigcup_{i=1}^{N} E_i \end{cases}$. "⊃" $\forall x\in X\quad f(x)\in\{v_1,\dots,v_N\}$.

Assume that v_i's are distinct. Then $f=\sum_{i=1}^{N} v_i c_{f^{-1}(\{v_i\})}$. Note that $\{v_i\}=[v_i,v_i]\in\mathscr{B}_{\mathbb{R}}$. f is measurable; hence, $f^{-1}\Big(\underbrace{\{v_i\}}_{\in\mathscr{B}_{\mathbb{R}}}\Big)\in\mathscr{F}$. Because v_i's are distinct, $f^{-1}\Big(\underbrace{\{v_i\}}_{\in\mathscr{B}_{\mathbb{R}}}\Big)$'s are disjoint.

$\Rightarrow$ f is measurable.

Proof. $E_i\in\mathscr{F}\ \Rightarrow\ c_{E_i}$ measurable $\Rightarrow\ f=\sum_{i=1}^{N} a_i c_{E_i}$ measurable.

- Not require a_i's to be distinct.
- f does not have a unique representation.

- Let $f:X\to\mathbb{R}$ measurable, $f\geq 0$. Then $\exists s_n:X\to\mathbb{R}\ \ s_n\geq 0$ simple function such that $s_n \nearrow f$ (s_n converges pointwise to f in a monotonic increasing manner.)

$$s_n(x)=\sum_{k=0}^{2^{2n}-1}\frac{k}{2^n}c_{f^{-1}\left(\left[\frac{k}{2^n},\frac{k+1}{2^n}\right)\right)}+2^n c_{f^{-1}([2^n,\infty))}$$

- Let $f:X\to\mathbb{R}$ measurable, then $\exists s_n:X\to\mathbb{R}$ simple function such that $\forall x\in X$ $s_n(x)\to f(x)$.

  - $f(x)=\underbrace{\max(f,0)}_{f^+}-\underbrace{\max(-f,0)}_{f^-}$
  - $f^+, f^-\geq 0$

$\int f\,dm$ for simple functions

- Let $(X,\mathscr{F},m)$ be a measure space, $f:X\to\mathbb{R}$ is a simple function. Then $\int f\,dm=\sum_{i=1}^{N} a_i m(E_i)$.

  - $\int f\,dm$ does not depend on the representation.

    $f=\sum_{i=1}^{N} a_i c_{E_i}=\sum_{i=1}^{M} b_i c_{F_i}$ $E_i\subset X$, $\in\mathscr{F}$, and disjoint, $F_i\subset X$, $\in\mathscr{F}$, and disjoint, $a_i,b_i\neq 0$

    $\Rightarrow \int f\,dm=\sum_{i=1}^{N} a_i m(E_i)=\sum_{j=1}^{M} b_j m(F_j)$.

- Let $f,g:X\to\mathbb{R}$, simple, $f\geq g\geq 0$, then $\int f\,dm\geq\int g\,dm$

- Let $(X,\mathcal{F})$ be a measurable space. $f: X \to \mathbb{R}$ is a simple function. Then,
  - $|f|$ is a simple function.

    Proof. $f = \sum_{i=1}^{N} a_i c_{E_i}$ where $E_i \subset X$, $\in \mathcal{F}$, and disjoint, $a_i \neq 0$. Let $g = |f|$.

    Then, g can be written as $g = \sum_{i=1}^{N} b_i c_{E_i}$ where $b_i = |a_i|$. $a_i \neq 0 \Rightarrow b_i = |a_i| \neq 0$. Also, we already have $E_i \subset X$, $\in \mathcal{F}$, and disjoint because f is a simple function.
  - $\left|\int f\,dm\right| \le \int |f|\,dm$

    Proof. $f = \sum_{i=1}^{N} a_i c_{E_i} \Rightarrow \int f dm = \sum_{i=1}^{N} a_i m(E_i)$, and $|f| = \sum_{i=1}^{N} |a_i|\, c_{E_i} \Rightarrow \int |f|\,dm = \sum_{i=1}^{N} |a_i| m(E_i)$. By triangle inequality, $\left|\sum_{i=1}^{N} a_i m(E_i)\right| \le \sum_{i=1}^{N} |a_i m(E_i)|$. Also, because $m(E_i) \ge 0$, $|a_i m(E_i)| = |a_i|\, m(E_i)$.

$\int f\,dm$

- **Lebesgue approximate sums**: Let $f: X \to \mathbb{R}$ measurable, $f \ge 0$, partition $\mathcal{P}_n = \left\{\left[0, \frac{1}{2^n}\right), \left[\frac{1}{2^n}, \frac{2}{2^n}\right), \ldots, \left[2^n - \frac{1}{2^n}, 2^n\right), [2^n, \infty)\right\}$. Then $L(f, \mathcal{P}_n) = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} m\left(f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)\right) + 2^n m\left(f^{-1}\left([2^n, \infty)\right)\right)$

  - $L(f, \mathcal{P}_n) = \int s_n\,dm$ where $s_n(x) = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} c_{f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)} + 2^n c_{f^{-1}([2^n, \infty))}$ simple, ≥ 0.
    - $s_n \nearrow f$.
  - Let $I_k = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$ for $k = 0, \ldots, 2^{2n-1}$. $I_{2^{2n}} = [2^n, \infty)$. Then
    - $s_n(x) = \sum_{k=0}^{2^{2n}} \inf(I_k)\, c_{f^{-1}(I_k)}$.
    - $L(f, \mathcal{P}_n) = \sum_{k=0}^{2^{2n}} \inf(I_k)\, m\left(f^{-1}(I_k)\right)$.
    - $\inf(I_k) = \frac{k}{2^n}$

- Let $(X, \mathcal{F}, m)$ be a measure space, $f: X \to \mathbb{R}$ measurable, $f \ge 0$, define

$\int f dm = \lim_{n\to\infty} L(f, \mathscr{P}_n) \in [0,\infty]$

- $\lim_{n\to\infty} L(f, \mathscr{P}_n)$ exists (allow $+\infty$)

  Proof. $L(f,\mathscr{P}_n) = \int s_n dm$. Because $s_n \nearrow f$, and $s_n \geq 0$, we have $\int s_n dm \leq \int s_{n+1} dm$.

  Thus, the sequence $\left(\int s_n dm\right)_{n=1}^{\infty}$ is a monotone increasing sequence of real number.

- Let $f : X \to \mathbb{R}$ simple $f \geq 0$. Then, $\lim_{n\to\infty} L(f,\mathscr{P}_n) = \int f dm$.

  - Let $f = \sum_{i=1}^{N} a_i c_{E_i}$ where $E_i \subset X$, $\in \mathscr{F}$, and disjoint, $a_i > 0$, then

    $\lim_{n\to\infty} L(f,\mathscr{P}_n) = \int f dm = \sum_{i=1}^{N} a_i m(E_i)$.

- Let $f, g : X \to \mathbb{R}$ measurable, $\forall x \in X$ $0 \leq f \leq g$, then

  - For $s_n^{(f)}(x) = \sum_{k=0}^{2^{2n}} \inf(I_k) c_{f^{-1}(I_k)}$, $s_n^{(g)}(x) = \sum_{k=0}^{2^{2n}} \inf(I_k) c_{g^{-1}(I_k)}$, we have $\forall x \in X$ $s_n^{(f)}(x) \geq s_n^{(g)}(x)$

    Proof. Consider any $x \in X$. Then $\exists k_0$ $f(x) \in I_{k_0} \Rightarrow s_n^{(f)}(x) = \inf\left(I_{k_0}\right)$, and $\exists k_1$ $g(x) \in I_{k_1} \Rightarrow s_n^{(g)}(x) = \inf\left(I_{k_1}\right)$. Because $f \leq g$, $k_0 \leq k_1 \Rightarrow \inf\left(I_{k_0}\right) \leq \inf\left(I_{k_1}\right)$.

    Hence, $s_n^{(f)}(x) = \inf\left(I_{k_0}\right) \leq \inf\left(I_{k_1}\right) = s_n^{(g)}(x)$.

  - $\int f dm \leq \int g dm$.

    Proof. $\forall x \in X$ $0 \leq s_n^{(f)}(x) \leq s_n^{(g)}(x) \Rightarrow L(f,\mathscr{P}_n) = \int s_n^{(f)} dm \leq \int s_n^{(g)} dm = L(g,\mathscr{P}_n)$.

    Take lim $n \to \infty$.

- Let $f_k : X \to \mathbb{R}$ measurable, $0 \leq f_1 \leq f_2 \leq \cdots$

  - Let $g : X \to \mathbb{R}$ simple function, $\forall x \in X$ $\lim_{k\to\infty} f_k(x) \geq g(x) \geq 0$, then $\lim_{k\to\infty} \int f_k dm \geq \int g dm$.
  - Let $\forall x \in X$ $\lim_{k\to\infty} f_k(x) \geq b c_B$, $b > 0$, $B \in \mathscr{F}$, then $\lim_{k\to\infty} \int f_k dm \geq b m(B)$.

- **The monotone convergence theorem**: Let $0 \leq f_1 \leq f_2 \leq \cdots$, $f_k : X \to \mathbb{R}$ measurable, $\forall x \in X$ $\lim_{k\to\infty} f_k(x) = f(x)$ $\left(f_k \nearrow f \text{ pointwise}\right)$, then $\lim_{k\to\infty} \int f_k dm = \int f dm$

  Proof. Because f_k's are measurable, $f = \lim_{k\to\infty} f_k$ is measurable. 1) $\lim_{k\to\infty} \int f_k dm \leq \int f dm$:

  Because f_k, f measurable, $\forall x \in X$ $0 \leq f_k \leq f$, we have $\int f_k dm \leq \int f dm$. Take lim as k

$\to \infty$. 2) $\lim_{k\to\infty} \int f_k dm \ge \int f dm$: $s_n \nearrow f$; so $\lim_{k\to\infty} f_k(x) = f(x) \ge s_n(x) \Rightarrow$ $\lim_{k\to\infty} \int f_k dm \ge \int s_n dm = L(f, \mathscr{P}_n)$. Take lime $n \to \infty$.