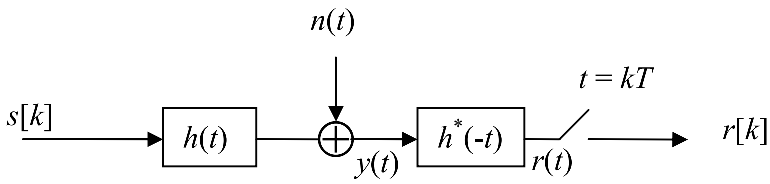
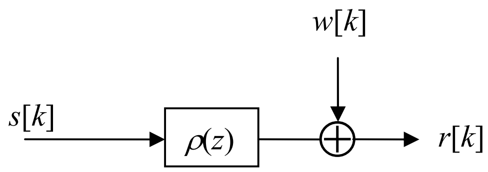
Suboptimal Receivers



- $r[k]$ give sufficient statistics.
- $r[k]$ satisfies the equivalent discrete-time model

  $$r[k]=\rho[k]*s[k]+w[k]\xrightarrow{z}r(z)=\rho(z)s(z)+w(z)$$

  - $w[k]=\left(n(t)*h^*(-t)\right)\Big|_{kT}$



  - $\rho(t)=h(t)*h^*(-t)=\int_{-\infty}^{\infty}h(\tau)h^*(\tau-t)d\tau$

  Proof

$$\begin{aligned}r(t)&=\left(\sum_{n=-\infty}^{\infty}s[n]h(t-nT)+n(t)\right)*h^*(-t)\\&=\left(\left(\sum_{n=-\infty}^{\infty}s[n]\delta(t-nT)\right)*h(t)+n(t)\right)*h^*(-t)\\&=\left(\sum_{n=-\infty}^{\infty}s[n]\delta(t-nT)\right)*h(t)*h^*(-t)+n(t)*h^*(-t)\\&=\left(\sum_{n=-\infty}^{\infty}s[n]\delta(t-nT)\right)*\rho(t)+n(t)*h^*(-t)\\&=\sum_{n=-\infty}^{\infty}s[n]\rho(t-nT)+n(t)*h^*(-t)\end{aligned}$$

$$\begin{aligned}r[k]=r(kT)&=\sum_{n=-\infty}^{\infty}s[n]\rho(kT-nT)+n(t)*h^*(-t)\Big|_{t=kT}\\&=\sum_{n=-\infty}^{\infty}s[n]\rho[k-n]+n(t)*h^*(-t)\Big|_{t=kT}\end{aligned}$$

  - $\rho[-k]=\rho^*[k]$, $\rho^{\#}(z)=\rho(z)$, $\rho(t)\overset{\mathcal{F}}{\rightleftharpoons}Q(f)=|H(f)|^2$.

- $\rho(t) \overset{\mathcal{F}}{\rightleftharpoons} Q(f) = |H(f)|^2$

  $$\rho[k] \xrightarrow{DTFT} Q_{DTFT}(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Q\left(\frac{f}{T}+\frac{n}{T}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn2\pi f}$$

  $$\rho[k] \xrightarrow{DTFT} Q_{DTFT}(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Q\left(\frac{\omega}{2\pi T}+\frac{n}{T}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}$$

  $\rho(t) \overset{\mathcal{F}}{\rightleftharpoons} \hat{Q}(\Omega) = |\hat{H}(\Omega)|^2$

  $$\rho[k] \xrightarrow{DTFT} Q_{DTFT}(\omega) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\hat{Q}\left(\frac{\omega}{T}+k\frac{2\pi}{T}\right)\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}$$

- If $n(t)$ is white with PSD N_0, then $R_{ww}[k] = N_0\rho[k] \xrightarrow{Z} S_{ww}(z) = N_0\rho(z)$. (Not white unless $\rho[k] = \delta[k]$.

  Proof Let $g(t) = h^*(-t)$. Because $w[k] = \left(n(t) * g(t)\right)\big|_{kT}$, we know that

  $$R_{ww}[k] = N_0\left(g(t) * g^*(-t)\right)\big|_{t=kT} = N_0\left(h^*(-t) * h(t)\right)\big|_{t=kT} = N_0\rho[k].$$

- The noise sequence $w[k]$ can be generated by a white noise with PSD N_0 and a linear stable causal monic phase filter $\beta(z)$.

  Proof We know that $\rho(z) = \rho^{\#}(z)$; thus, its poles and zeroes are located symmetrically with respect to the unit circle. We can then factorize $\rho(z)$ into $\gamma\beta(z)\beta^{\#}(z)$ where the poles and zeroes of $\beta(z)$ are the poles and zeroes of $\rho(z)$ that are located inside the unit circle. By choosing the ROC of $\beta(z)$ such that it includes the unit circle and extends outward, $\beta(z)$ is stable, causal, and minimum phase.

  Now, if we filter white noise $n(z)$ with PSD N_0 with $\beta(z)$, the output will be $N_0\beta(z)\beta^{\#}(z)$.

  To create white noise with PSD γ, we can scale white noise with PSD N_0 by $\sqrt{\frac{\gamma}{N_0}}$.

Linear algebra

- $\langle y[\cdot], x[\cdot]\rangle = \sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

- $\|x[\cdot]\|^2 = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

Correlation and Correlation Spectrum

- **Z-tranform**: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$; $0 \le R_a < |z| < R_b \le \infty$

For $R_a < \hat{R} < R_b$, $x[n] = \frac{1}{2\pi}\int_0^{2\pi} X\left(\hat{R}e^{j\theta}\right)\hat{R}^n e^{jn\theta}d\theta$

- $x[k] \xrightarrow{Z} X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k} \xrightarrow{z=e^{j\omega}} X\left(e^{j\omega}\right)$
- $x^*[-k] \xrightarrow{Z} X^{\#}(z) = \sum_{k=-\infty}^{\infty} x^*[-k]z^{-k} = X^*\left(\frac{1}{z^*}\right) \xrightarrow{z=e^{j\omega}} X^*\left(e^{j\omega}\right)$

Proof $x^*[-k] \xrightarrow{Z} X^{\#}(z) = \sum_{k=-\infty}^{\infty} x^*[-k]z^{-k} = \left(\sum_{k=-\infty}^{\infty} x[-k]\left(z^*\right)^{-k}\right)^*$

$$= \left(\sum_{n=-\infty}^{\infty} x[n]\left(\frac{1}{z^*}\right)^{-n}\right)^* = X^*\left(\frac{1}{z^*}\right)$$

- $\left(X^{\#}(z)\right)^{\#} = X(z)$

  $\left(X(z)Y(z)\right)^{\#} = X^{\#}(z)Y^{\#}(z)$

  $\left(X(z)+Y(z)\right)^{\#} = X^{\#}(z)+Y^{\#}(z)$

Proof $\left(X^{\#}(z)\right)^{\#} = \left(X^*\left(\frac{1}{\left(\frac{1}{z^*}\right)^*}\right)\right)^* = X(z)$

$$\left(X(z)Y(z)\right)^{\#} = \left(X\left(\frac{1}{z^*}\right)Y\left(\frac{1}{z^*}\right)\right)^* = X^*\left(\frac{1}{z^*}\right)Y^*\left(\frac{1}{z^*}\right) = X^{\#}(z)Y^{\#}(z)$$

$$\left(X(z)+Y(z)\right)^{\#} = \left(X\left(\frac{1}{z^*}\right)+Y\left(\frac{1}{z^*}\right)\right)^* = X^*\left(\frac{1}{z^*}\right)+Y^*\left(\frac{1}{z^*}\right) = X^{\#}(z)+Y^{\#}(z)$$

- $x[k] = x^*[-k] \Rightarrow X(z) = X^{\#}(z)$
- $X^{\#}\left(\frac{1}{z_0^*}\right) = X^*(z_0)$ and $X^{\#}(z_1) = X^*\left(\frac{1}{z_1^*}\right)$

Proof $X^{\#}\left(\frac{1}{z_0^*}\right) = X^*\left(\frac{1}{\left(\frac{1}{z_0^*}\right)^*}\right) = X^*(z_0)$. Substituting $z_0 = \frac{1}{z_1^*}$, we have $X^{\#}(z_1) = X^*\left(\frac{1}{z_1^*}\right)$.

- $X(z_0) = 0 \Rightarrow X^{\#}\left(\frac{1}{z_0^*}\right) = 0$. $X^{\#}(z_1) = 0 \Rightarrow X\left(\frac{1}{z_1^*}\right) = 0$.

Proof $X^{\#}\left(\frac{1}{z_0^*}\right)=X^*\left(z_0\right)=0^*=0$.

- $\left|\lim_{z\to z_0} X(z)\right|=\infty \Rightarrow \left|\lim_{z\to \frac{1}{z_0^*}} X^{\#}(z)\right|=\infty$. $\left|\lim_{z\to z_1} X^{\#}(z)\right|=\infty \Rightarrow \left|\lim_{z\to \frac{1}{z_1^*}} X(z)\right|=\infty$.

  Proof Use $X^{\#}\left(\frac{1}{z_0^*}\right)=X^*\left(z_0\right)$.

- If $X(z)=X^{\#}(z)$, then its poles and zeroes are located symmetrically with respect to the unit circle.

  Proof $X\left(z_0\right)=0 \Rightarrow X^{\#}\left(\frac{1}{z_0^*}\right)=0$. Thus, $X\left(\frac{1}{z_0^*}\right)=X^{\#}\left(\frac{1}{z_0^*}\right)=0$.

  $\left|\lim_{z\to z_0} X(z)\right|=\infty \Rightarrow \left|\lim_{z\to z_0} X^{\#}\left(\frac{1}{z^*}\right)\right|=\infty$. Thus, $\left|\lim_{z\to z_0} X\left(\frac{1}{z^*}\right)\right|=\left|\lim_{z\to z_0} X^{\#}\left(\frac{1}{z^*}\right)\right|=\infty$.

  Note also that $\frac{1}{z_0^*}=\frac{1}{\left|z_0\right|}\left(\frac{z_0}{\left|z_0\right|}\right)$; thus $\angle\left(\frac{1}{z_0^*}\right)=\angle z_0$ and $\left|\frac{1}{z_0^*}\right|=\frac{1}{\left|z_0\right|}$. So, $\frac{1}{z_0^*}$ is along the same line from origin as z_0, but its magnitude is $\frac{1}{\left|z_0\right|}$. Also, $\left|\frac{1}{z_0^*}\right|>1 \Leftrightarrow \left|z_0\right|<1$ and $\left|\frac{1}{z_0^*}\right|<1 \Leftrightarrow \left|z_0\right|>1$.

- If $g(z)$ is monic and causal, then $g^{\#}(z)$ is monic and anti-causal.

  Proof $g(z)$ is monic and causal; thus, it can be written in the form $g(z)=1+\sum_{k=1}^{\infty} a_k z^{-k}$.

  $g^{\#}(z)=g^*\left(\frac{1}{z^*}\right)=1+\left(\sum_{k=1}^{\infty} a_k\left(z^*\right)^k\right)^*=1+\sum_{k=1}^{\infty} a_k z^k$. Thus, $g^{\#}(z)$ is monic and anticausal.

- If $g(z)$ is SCAMP, then $g^{-1}(z)$ is also SCAMP.

  Proof $g(z)$ has all of its zeros and poles inside the unit circle. Now, the zeros of $g^{-1}(z)$ are the poles of $g(z)$ and vice versa. Thus, zeros and poles of $g^{-1}(z)$ are also inside the unit circle.

  So, if we define the ROC if $g^{-1}(z)$ to be exactly outside the outermost pole, then $g^{-1}(z)$ is causal, stable, and minimum phase.

  Because $g(z)$ is monic, $\lim_{|z|\to\infty} g(z)=1$. Therefore, $\lim_{|z|\to\infty} g^{-1}(z)=\frac{1}{1}=1$, which implies that $g^{-1}(z)$ is also monic.

- If $g(z)$ is SCAMP, then $g^{-1}(z)$ is monic, anticausal, stable, and maximum phase.

  Proof $g(z)$ is monic and causal; thus $g^{\#}(z)$ is monic and anti-causal.

z_0 is a zero/pole of $g(z)$ if and only if $\frac{1}{z_0^*}$ is a zero/pole of $g^\#(z)$. Thus, All poles and zeros of $g^\#(z)$ are outside the unit circle. So, $g^\#(z)$ is maximum phase. If de define the ROC of $g^{-1}(z)$ to be the disc inside the innermost pole, then $g^\#(z)$ is stable (and anti-causal.)

- $x[k]*y^*[-k]=\sum_{n=-\infty}^{\infty} x[n]y^*[n-k] \xrightarrow{Z} X(z)Y^\#(z)=\sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} x[n]y^*[n-k]\right)z^{-k}$.
- Given WSS $x[n], y[n]$

  $R_{xy}[k]=E\left[x[n]y^*[n-k]\right] \xrightarrow{Z} S_{xy}(z) \overset{?}{=} \hat{E}\left[X(z)Y^\#(z)\right]$ power spectrum density

  $R_{xy}[k]=\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xy}\left(e^{j\omega}\right)e^{jk\omega}d\omega \overset{DTFT}{\rightleftharpoons} S_{xy}\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} R_{xy}[n]e^{-jn\omega}$.
- $R_{yx}^*[-k]=R_{xy}[k]$.

  Proof $R_{yx}^*[-k]=\left(E\left[y[n]x^*[n+k]\right]\right)^*=E\left[y^*[n]x[n+k]\right] \underset{WSS}{=} E\left[y^*[n'-k]x[n']\right]=R_{xy}[k]$.
- $S_{yx}^\#(z)=S_{xy}(z)$

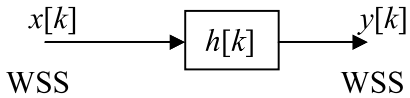
  Proof Because $R_{yx}^*[-k] \xrightarrow{Z} S_{yx}^\#(z)$, and $R_{xy}[k] \xrightarrow{Z} S_{xy}(z)$, the equality $R_{yx}^*[-k]=R_{xy}[k]$ implies $S_{yx}^\#(z)=S_{xy}(z)$.
- $R_{XX}[-k]=R_{XX}^*[k]$, $S_{XX}^\#(z)=S_{XX}(z)$.
- $S_{XX}\left(e^{j\omega}\right)$ is real

  Proof From $S_{XX}^\#(z)=S_{XX}(z)$, we have $S_{XX}^\#\left(e^{j\omega}\right)=S_{XX}\left(e^{j\omega}\right)$.

  By definition, we have $S_{XX}^\#\left(e^{j\omega}\right)=S_{XX}^*\left(\frac{1}{\left(e^{j\omega}\right)^*}\right)=S_{XX}^*\left(e^{j\omega}\right)$.

  Thus, $S_{XX}^*\left(e^{j\omega}\right)=S_{XX}\left(e^{j\omega}\right)$.
- White process $x[k] \Rightarrow S_{xx}(z)=E_x=E\left[\left|x[n]\right|^2\right]$
- $R_{xx}[0]=E\left[\left|x[k]\right|^2\right]=\int_{2\pi}\left|S_{xx}\left(e^{j\omega}\right)\right|^2 d\omega$.

  Proof $R_{xx}[0]=\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}\left(e^{j\omega}\right)e^{j0\omega}d\omega=\frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}\left(e^{j\omega}\right)d\omega$.
- Linear filtering of a WSS process



- $R_Y[k] = R_X[k] * h[k] * h^*[-k]$

  Proof $\quad y[m] = x[m] * h[m] = \sum_{n=-\infty}^{\infty} x[m-n]h[n]$

$$y[m-k] = \sum_{n'=-\infty}^{\infty} x[m-k-n']h[n']$$

$$y^*[m-k] = \sum_{n'=-\infty}^{\infty} x^*[m-k-n']h^*[n']$$

$$R_Y[k] = E\left[y[m]y^*[m-k]\right] = E\left[\left(\sum_{n=-\infty}^{\infty} x[m-n]h[n]\right)\left(\sum_{n'=-\infty}^{\infty} x^*[m-k-n']h^*[n']\right)\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} E\left[x[m-n]x^*[m-k-n']\right]h[n]h^*[n']$$

$$\sum_{n=-\infty}^{\infty}\sum_{n'=-\infty}^{\infty} R_X[k+n'-n]h[n]h^*[n']$$

$$R_X[k] * h[k] * h^*[-k] = \left(\sum_{n=-\infty}^{\infty} R_X[k-n]h[n]\right) * h^*[-k]$$

$$= \sum_{n'=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} R_X[k-n'-n]h[n]h^*[-n']$$

$$= \sum_{n'=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} R_X[k+n''-n]h[n]h^*[n'']$$

- $S_{yy}(z) = H(z)H^{\#}(z)S_{xx}(z)$

  Proof $\;$ Directly from $R_Y[k] = R_X[k] * h[k] * h^*[-k]$.

- $S_{yy}(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 S_{xx}(e^{j\omega})$

- For an FIR filter with white input $\left(S_{xx}(z) = E_x = E\left[\left|x[n]\right|^2\right]\right)$,
  - $S_{yy}(z) = E_x H(z)H^{\#}(z)$
  - The PSD of the output has zeros symmetrical around the unit circle.

    Proof $\;$ Because $H(z_0) = 0 \Leftrightarrow H^{\#}\left(\frac{1}{z_0^*}\right) = 0$. Thus, $S_{yy}(z_0) = 0 \Leftrightarrow S_{yy}\left(\frac{1}{z_0^*}\right) = 0$.

- Filter $h[k]$

- An LTI filter $H(z)=\frac{B(z)}{A(z)}$ is **minimum phase**
  - if all its poles and zeros are inside the unit circles $|z|=1$.

  $\equiv$ if both it and its inverse $\frac{1}{H(z)}$ are stable.
  - $\Rightarrow$ causal.
- A polynomial of the form $H(z)=h_0+h_1z^{-1}+\ldots+h_Nz^{-N}=h_0\left(1-z_1z^{-1}\right)\left(1-z_2z^{-1}\right)\cdots\left(1-z_Nz^{-1}\right)$ is minimum phase if all of its roots z_i are inside the unit circle, i.e. $|z_i|<1$.
- **Monic** if $h[0]=1$.
- Causal $\Rightarrow$ [monic $\Leftrightarrow \lim_{|z|\to\infty} H(z)=1$]
- If rational $H(z)$, minimum phase if all its zeros are inside the unit circle.
- For a causal minimum phase $h[k]$, the partial energy $\mathcal{E}[n]=\sum_{k=0}^{n}|h[k]|^2$ is maximum for all n among all sequences with the same energy. Thus, minimum-phase signals are maximally concentrated toward time 0 among the space of causal signals for a given magnitude spectrum.
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- **Spectral Factorization**
  - Let $x[k]$ be a stationary process with power spectrum density $S_{xx}(z)=E\left[X(z)X^{\#}(z)\right]$.

    If $\int_{-\pi}^{\pi}\ln S_{xx}\left(e^{j\omega}\right)d\omega>-\infty$, then there exists a $\gamma>0$ and a unique stable, causal, monic, and minimum phase (SCAMP) filter $g[k]\xrightarrow{Z}G(z)$ such that $S_{xx}(z)=\gamma G(z)G^{\#}(z)$, $\gamma=e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln S_{xx}\left(e^{j\omega}\right)d\omega}$.
    - $G^{-1}(z)$ is also SCAMP.
  - Given $S_{xx}(z)$, we can generate $x[k]$ (in fact, generate sth. that has the same spectrum as $x[k]$) using a linear filter $G(z)$ with white input.
- Whittle's Construction:

  If $\ln S_{xx}(z)$ is analytic in $\rho<|z|<\frac{1}{\rho}$ $(\rho<1)$, then it has the Laurent expansion $\ln S_{xx}(z)=\sum_{k=-\infty}^{\infty}c_kz^{-k}$

  where $c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln S_{xx}\left(e^{j\omega}\right)e^{jk\omega}d\omega\xrightarrow{\text{DTFT}}\ln S_{xx}\left(e^{j\omega}\right)$. c_k is real and even. (??)

  Then $S_{xx}(z)=e^{\sum_{-\infty}^{\infty}c_kz^{-k}}=e^{\sum_{k=-\infty}^{-1}c_kz^{-k}}e^{c_0}e^{\sum_{k=1}^{\infty}c_{-k}z^{-k}}=e^{c_0}e^{\sum_{k=1}^{\infty}c_kz^{-k}}e^{\sum_{k=1}^{\infty}c_{-k}z^{k}}$.

  Define

$$\gamma = e^{c_0} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{xx}\left(e^{j\omega}\right) d\omega}.$$

$G(z) = e^{\sum_{k=1}^{\infty} c_k z^{-k}}$. Then $G^{\#}(z) = G^{*}\left(\frac{1}{z^{*}}\right) = \left(e^{\sum_{k=1}^{\infty} c_k \left(\frac{1}{z^{*}}\right)^{-k}}\right)^{*} = e^{\sum_{k=1}^{\infty} c_k^{*} z^{k}} = e^{\sum_{k=1}^{\infty} c_k z^{k}} = e^{\sum_{k=1}^{\infty} c_{-k} z^{k}}$.

- $G(z)$ is causal (involving only z^{-k} ??)
- $G(z)$ is monic because $\lim_{|z|\to\infty} G(z) = 1$.
- $G(z)$ is minimum phase because $G^{-1}(z)$ is also analytical (??)

- The Whitening Filter:

  Suppose that $x[k]$ is a WSS process with PSD that admits the spectral factorization $S_{xx}(z) = \gamma G(z) G^{\#}(z)$,
  $\gamma = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{xx}\left(e^{j\omega}\right) d\omega}$.

  If $x[k]$ be the input of a SCAMP filter, then, among the set of SCAMP filters, the whitening filter $\alpha(z) = \frac{1}{G(z)}$ minimizes the output variance. In particular, the output $y[k]$ is a white sequence and

  $$E\left[\left|y[k]\right|^2\right] = \gamma = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{xx}\left(e^{j\omega}\right) d\omega}.$$

  Proof For any causal, stable, and monic filter $\alpha(z)$ we have

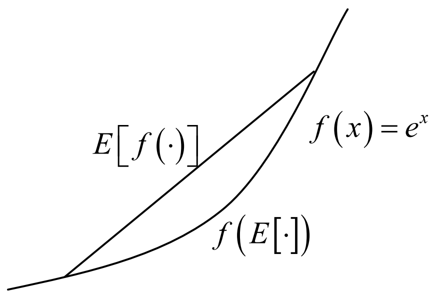
  $$\begin{aligned} S_{yy}(z) &= \alpha(z)\alpha^{\#}(z) S_{xx}(z) = \alpha(z)\alpha^{\#}(z)\gamma G(z) G^{\#}(z) \\ &= \gamma \alpha(z) G(z) \alpha^{\#}(z) G^{\#}(z) \\ &= \gamma \beta(z) \beta^{\#}(z) \end{aligned}$$

  where $\beta(z) = \alpha(z) G(z)$ is causal, stable, and monic.

  $$\gamma = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{xx}\left(e^{j\omega}\right) d\omega} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{yy}\left(e^{j\omega}\right) d\omega}.$$

  Applying Jensen's inequality,

  $$\gamma = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{yy}\left(e^{j\omega}\right) d\omega} \le \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\ln S_{yy}\left(e^{j\omega}\right)} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}\left(e^{j\omega}\right) d\omega = E\left[\left|y[k]\right|^2\right].$$



Equality holds if $S_{yy}\left(e^{j\omega}\right)=\gamma$. (Equality holds if $S_{yy}\left(e^{j\omega}\right)$ is constant. Since $\beta(z)=\alpha(z)G(z)$ is monic, it is a constant iff $\beta(z)=\alpha(z)G(z)=1$)

This happens when $\alpha(z)=\dfrac{1}{G(z)}$.

Optimal Linear Prediction

- Given $x[t-1]$, $x[t-2]$, …,
  find a stable linear predictor $A(z)$ (causal, $a_0 = 0$)

$$\hat{x}[t]=\sum_{k=1}^{\infty}a_k x[t-k]\xrightarrow{Z}\hat{X}(z)=A(z)X(z)$$

$$e[t]=x[t]-\hat{x}[t]\xrightarrow{Z}X(z)\left(1-A(z)\right)=X(z)\alpha(z);\ \alpha(z)=1-A(z).$$

  such that the mean square error $\mathcal{E}=E\left[\left|e[t]\right|^2\right]$ is minimized by a causal monic and stable $\alpha(z)$.

- If $S_{xx}(z)=\gamma\beta(z)\beta^{\#}(z)$, we obtain $A_{opt}(z)=1-\dfrac{1}{\beta(z)}$

  Proof Note that $\alpha(z)=1-A(z)$ is causal, stable, and monic.

  Because $e(z)=X(z)\alpha(z)$, by the whitening filter theorem, the whitening filter $\alpha(z)=\dfrac{1}{\beta(z)}$ minimizes the output variance, which is $\mathcal{E}=E\left[\left|e[t]\right|^2\right]$. From $\alpha(z)=1-A(z)$, we have

$$A_{opt}(z)=1-\frac{1}{\beta(z)}.$$

MMSE Linear Equalizer

- Scalar.
  Consider two zero mean random variable s and y. Knowing y, we like to infer s by exploiting the joint distribution of s and y.
  $\hat{s}=fy$.
  $e=s-fy$. Want to minimize $E\left[|e|^2\right]=E\left[|s-fy|^2\right]$.
  - Completing the squares solution:

$$\begin{aligned}\mathcal{E}&=E\left[|\hat{s}-s|^2\right]=E\left[|fy-s|^2\right]=E\left[(fy-s)\left(f^*y^*-s^*\right)\right]\\&=|f|^2E\left[|y|^2\right]+E\left[|s|^2\right]-fE\left[ys^*\right]-f^*E\left[y^*s\right]\\&=|f|^2R_{yy}+R_{ss}-fR_{ys}-f^*R_{sy}\end{aligned}$$

Consider $\left(f - R_{sy}R_{yy}^{-1}\right)R_{yy}\left(f - R_{sy}R_{yy}^{-1}\right)^* = \left(f - R_{sy}R_{yy}^{-1}\right)R_{yy}\left(f^* - R_{sy}^*\left(R_{yy}^{-1}\right)^*\right)$

$$= \left(fR_{yy} - R_{sy}\right)\left(f^* - \left(R_{yy}^{-1}\right)^* R_{ys}\right)$$

$$= |f|^2 R_{yy} - R_{sy}f^* - fR_{yy}\left(R_{yy}^{-1}\right)^* R_{ys} + R_{sy}\left(R_{yy}^{-1}\right)^* R_{ys}$$

$$= |f|^2 R_{yy} - R_{sy}f^* - fR_{ys} + R_{sy}\left(R_{yy}^{-1}\right)^* R_{ys}$$

$$= \mathcal{E} - R_{ss} + R_{sy}\left(R_{yy}^{-1}\right)^* R_{ys}$$

So, $f_{MMSE} = R_{sy}R_{yy}^{-1}$ and $\mathcal{E}_{MMSE} = R_{ss} - R_{sy}\left(R_{yy}^{-1}\right)^* R_{ys}$.

- Geometrical solution:
  Want to find $\hat{s} = fy$ closet to s in the direction of y.

  The necessary and sufficient condition of optimality is $e \perp y \equiv E\left[ey^*\right] = 0$.

  $E\left[(s - fy)y^*\right] = E\left[sy^*\right] - fE\left[yy^*\right] = R_{sy} - fR_{yy} = 0$.

  Thus, $f_{MMSE} = R_{sy}R_{yy}^{-1}$.

  $$E\left[\mathcal{E}_{MMSE}\right] = E\left[|e|^2\right] = E\left[|s|^2\right] - E\left[|f_{MMSE}y|^2\right] = R_{ss} - E\left[\left|R_{sy}R_{yy}^{-1}y\right|^2\right]$$

  $$= R_{ss} - R_{sy}R_{yy}^{-1}R_{sy}^*R_{yy}^{-1}R_{yy} = R_{ss} - R_{sy}R_{yy}^{-1}R_{ys}$$

- R_{yy} is real.

- Vectors: Estimating a WSS random process $s[k]$ by linear-filtering a WSS process $y[k]$.
  $\hat{s}(z) = f(z)y(z)$.

  Define $e(z) = \hat{s}(z) - s(z)$. Want to minimize $S_{ee}(z) = E\left[e(z)e^{\#}(z)\right]$.

  Applying the orthogonal principle, we want $e(z) \perp y(z)$, i.e. $E\left[e(z)y^{\#}(z)\right] = 0$.

  From $E\left[e(z)y^{\#}(z)\right] = E\left[\left(\hat{s}(z) - s(z)\right)y^{\#}(z)\right] = E\left[\left(f(z)y(z) - s(z)\right)y^{\#}(z)\right]$

  $$= f(z)E\left[y(z)y^{\#}(z)\right] - E\left[s(z)y^{\#}(z)\right]$$

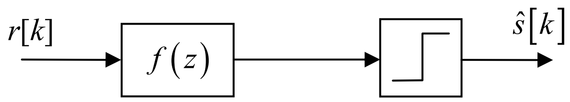
  $$= f(z)S_{yy}(z) - S_{sy}(z)$$

  We need $f_{opt}(z)S_{yy}(z) - S_{sy}(z) = 0$. Hence $f_{opt}(z) = \dfrac{S_{sy}(z)}{S_{yy}(z)}$.

$$
\begin{aligned}
S_{ee}(z) &= E\left[e(z)e^{\#}(z)\right] = E\left[\left(\frac{S_{sy}(z)}{S_{yy}(z)}y(z)-s(z)\right)\left(\frac{S_{sy}^{\#}(z)}{S_{yy}(z)}y^{\#}(z)-s^{\#}(z)\right)\right] \\
&= E\left[e(z)e^{\#}(z)\right] = E\left[\left(\frac{S_{sy}(z)}{S_{yy}(z)}y(z)-s(z)\right)\left(\frac{S_{ys}(z)}{S_{yy}(z)}y^{\#}(z)-s^{\#}(z)\right)\right] \\
&= E\left[\frac{S_{sy}(z)}{S_{yy}(z)}y(z)\frac{S_{ys}(z)}{S_{yy}(z)}y^{\#}(z)-s(z)\frac{S_{ys}(z)}{S_{yy}(z)}y^{\#}(z)-\frac{S_{sy}(z)}{S_{yy}(z)}y(z)s^{\#}(z)+s(z)s^{\#}(z)\right] \\
&= \frac{S_{sy}(z)}{S_{yy}(z)}\frac{S_{ys}(z)}{S_{yy}(z)}S_{yy}(z)-\frac{S_{ys}(z)}{S_{yy}(z)}S_{sy}(z)-\frac{S_{sy}(z)}{S_{yy}(z)}S_{ys}(z)+S_{ss}(z) \\
&= S_{ss}(z)-\frac{S_{sy}(z)}{S_{yy}(z)}S_{ys}(z)
\end{aligned}
$$

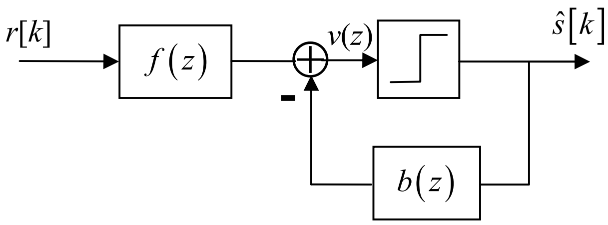
MMSE – Minimum Mean Square Error Estimation

MMSE Linear Equalizer



- $f(z)$: stable IIR, can be non-causal.

MMSE-DFE



- $f(z)$: stable IIR, can be non-causal.
- Causal $b(z)$, $b[0]=0$ $\left(\lim_{|z|\to\infty} b(z)=0\right)$. $b(z)=b_1z^{-1}+b_2z^{-2}+\cdots$
- If do optimally, should be better than MMSE without DFE because $b(z)=0$ case is included, so the optimal case should be better.
- Criterion: $\min_{f(z),b(z)} E\left[\left|v[k]-s[k]\right|^2\right]$.
- Assumption: $\hat{s}[k]\approx s[k]$.
- $$v(z)=r(z)f(z)-b(z)\hat{s}(z)\approx r(z)f(z)-b(z)s(z)$$

$$e(z)=v(z)-s(z)=r(z)f(z)-b(z)s(z)-s(z)=r(z)f(z)-(b(z)+1)s(z).$$
$$=r(z)f(z)-\beta(z)s(z)$$

Note that $\beta(z)=b(z)+1$ is IIR, causal, stable, and monic.

Applying the orthogonal principle, we need $e(z)\perp r(z)$,

$$E\left[\left(r(z)f(z)-\beta(z)s(z)\right)r^{\#}(z)\right]=f(z)S_{rr}(z)-\beta(z)S_{sr}(z)=0$$

$$f(z)=\frac{S_{sr}(z)}{S_{rr}(z)}\beta(z)$$

$$e(z)=r(z)f(z)-\beta(z)s(z)=r(z)\frac{S_{sr}(z)}{S_{rr}(z)}\beta(z)-\beta(z)s(z)$$
$$=\beta(z)\left(\frac{S_{sr}(z)}{S_{rr}(z)}r(z)-s(z)\right)$$
$$=\beta(z)\tilde{e}(z)$$

where $\tilde{e}(z)$ is the MSE of the linear MMSE estimator (without DFE).

The optimal feedback filter is given by choosing a SCAMP $\beta(z)$ to minimize $E\left[\left|e(z)\right|^2\right]$.

So, get $\beta(z)$ from spectral factorization of $S_{\tilde{e}\tilde{e}}(z)=S_{ss}(z)-\frac{S_{sy}(z)}{S_{yy}(z)}S_{ys}(z)$.

Discrete-time Equivalent Channel with MMSE

- $r[k]=\rho[k]*s[k]+w[k]\xrightarrow{z}r(z)=\rho(z)s(z)+w(z)$

  $R_{ww}[k]=N_0\rho[k]\xrightarrow{Z}S_{ww}(z)=N_0\rho(z)$

  $S_{ss}(z)=E_s$.
- $S_{rr}(z)=\rho(z)\left(\rho(z)S_{ss}(z)+N_0\right)$

  Proof From $r(z)=\rho(z)s(z)+w(z)$,

$$S_{rr}(z)=E\left[r(z)r^{\#}(z)\right]=E\left[\left(\rho(z)s(z)+w(z)\right)\left(\rho(z)s(z)+w(z)\right)^{\#}\right]$$
$$=E\left[\left(\rho(z)s(z)+w(z)\right)\left(\rho^{\#}(z)s^{\#}(z)+w^{\#}(z)\right)\right]$$
$$=E\left[\left(\rho(z)s(z)+w(z)\right)\left(\rho(z)s^{\#}(z)+w^{\#}(z)\right)\right]$$
$$=E\left[\rho(z)s(z)\rho(z)s^{\#}(z)+w(z)\rho(z)s^{\#}(z)+\rho(z)s(z)w^{\#}(z)+w(z)w^{\#}(z)\right]$$
$$=\rho(z)\rho(z)S_{ss}(z)+\rho^{\#}(z)E\left[s^{\#}(z)w(z)\right]+\rho(z)E\left[s(z)w^{\#}(z)\right]+S_{ww}(z)$$

From independence of $w[k]$ and $s[k]$, and assuming that the noise is zero mean,

$$S_{rr}(z) = \rho(z)\rho(z)S_{ss}(z) + \rho^{\#}(z)E\left[s^{\#}(z)\right]\cancelto{0}{E\left[w(z)\right]} + \rho(z)E\left[s(z)\right]\cancelto{0}{E\left[w^{\#}(z)\right]} + S_{ww}(z)$$

$$= \rho(z)\rho(z)S_{ss}(z) + S_{ww}(z)$$

We have shown that $S_{ww}(z) = \rho(z)N_0$; thus, $S_{rr}(z) = \rho(z)\left(\rho(z)S_{ss}(z) + N_0\right)$.

- Linear MMSE
  - $f_{MMSE}(z) = \dfrac{E_s}{\rho(z)E_s + N_0}$

    Proof $S_{sr}(z) = E\left[s(z)r^{\#}(z)\right] = E\left[s(z)\left(\rho(z)s(z) + w(z)\right)^{\#}\right] = E\left[s(z)\left(\rho(z)s^{\#}(z) + w^{\#}(z)\right)\right]$

    $$= \rho(z)S_{ss}(z) + E\left[s(z)w^{\#}(z)\right]$$

    From independence of *s* and *w*, we have $E\left[s(z)w^{\#}(z)\right] = E\left[s(z)\right]\cancelto{0}{E\left[w^{\#}(z)\right]} = 0$.

    Therefore, $S_{sy}(z) = \rho^{\#}(z)S_{ss}(z) = \rho(z)S_{ss}(z)$.

    $S_{rr}(z) = \rho(z)\left(\rho(z)S_{ss}(z) + N_0\right)$

    $$f_{MMSE}(z) = \frac{S_{sr}(z)}{S_{rr}(z)} = \frac{\rho(z)S_{ss}(z)}{\rho(z)\left(\rho(z)S_{ss}(z) + N_0\right)} = \frac{S_{ss}(z)}{\rho(z)S_{ss}(z) + N_0}.$$

    If we assume that *s*[*k*] is white, then $S_{ss}(z) = E_s$ and $f_{MMSE}(z) = \dfrac{E_s}{\rho(z)E_s + N_0}$.

  - $S_{ee}(z) = \dfrac{N_0E_s}{\rho(z)E_s + N_0}$

    Proof $S_{ee}(z) = S_{ss}(z) - \dfrac{S_{sr}(z)}{S_{rr}(z)}S_{rs}(z) = S_{ss}(z) - \dfrac{\rho(z)S_{ss}(z)\left(\rho(z)S_{ss}(z)\right)^{\#}}{\rho(z)\left(\rho(z)S_{ss}(z) + N_0\right)}$

    $$= S_{ss}(z) - \frac{\rho(z)S_{ss}^2(z)}{\rho(z)S_{ss}(z) + N_0} = \frac{N_0S_{ss}(z)}{\rho(z)S_{ss}(z) + N_0}$$

    If we assume that *s*[*k*] is white, then $S_{ss}(z) = E_s$ and $S_{ee}(z) = \dfrac{N_0E_s}{\rho(z)E_s + N_0}$.

  - $\varepsilon_{L-MMSE} = E\left[\left|e[k]\right|^2\right] = E_s\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\dfrac{1}{\rho\left(e^{j\omega}\right)\dfrac{E_s}{N_0} + 1}d\omega$

Proof $E\left[\left|e[k]\right|^2\right] = R_{ee}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{ee}\left(e^{j\omega}\right)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{N_0 E_s}{\rho\left(e^{j\omega}\right)E_s + N_0}d\omega$

$$= E_s\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{N_0}{\rho\left(e^{j\omega}\right)E_s + N_0}d\omega = E_s\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1}d\omega$$

- Define $SNR_{L-MMSE} = \frac{E_s}{\varepsilon_{L-MMSE}}$. Then $SNR_{L-MMSE} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1}d\omega\right)^{-1}$.

Proof $SNR_{L-MMSE} = \frac{E_s}{\varepsilon_{L-MMSE}} = \frac{E_s}{E_s\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1}d\omega} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1}d\omega\right)^{-1}$.

- MMSE-DFE
  - If $S_o(z) = \rho(z)E_s + N_0 = \gamma_0 g_0(z)g_0^{\#}(z)$, then $\beta(z) = g_0(z)$.

    Proof Consider $S_o(z) = \rho(z)E_s + N_0$.

    Let $S_o(z) = \gamma_0 g_0(z)g_0^{\#}(z)$. $S_{\tilde{e}\tilde{e}}(z) = \frac{N_0E_s}{\rho(z)E_s + N_0} = \frac{N_0E_s}{\gamma_0 g_0(z)g_0^{\#}(z)} = \frac{N_0E_s}{\gamma_0}\frac{1}{g_0(z)}\frac{1}{g_0^{\#}(z)}$

    Because $e(z) = \beta(z)\tilde{e}(z)$, to minimize $E\left[\left|e(z)\right|^2\right]$, $\beta(z) = \frac{1}{\frac{1}{g_0(z)}} = g_0(z)$. Then, minimum

    $E\left[\left|e(z)\right|^2\right] = \frac{N_0E_s}{\gamma_0}$.

  - $\gamma_0 = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln S_0\left(e^{j\omega}\right)d\omega} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho\left(e^{j\omega}\right)E_s + N_0\right)d\omega} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln N_0\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln N_0 + \ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}$

    $= N_0 e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}$

  - $f(z) = f_{L-MMSE}(z)\beta(z) = \frac{E_s}{\rho(z)E_s + N_0}\beta(z) = \frac{E_s g_0(z)}{\rho(z)E_s + N_0}$
  - $b_{MMSE-DFE}(z) = \beta(z) - 1 = g_0(z) - 1$.
  - $\varepsilon_{MMSE-DFE} = \frac{E_s}{e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}}$

Proof $\mathcal{E}_{MMSE-DFE} = \dfrac{N_0 E_s}{\gamma_0} = \dfrac{N_0 E_s}{N_0 e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}} = \dfrac{E_s}{e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}}$

- Define $SNR_{MMSE-DFE} = \dfrac{E_s}{\mathcal{E}_{MMSE-DFE}}$. Then $SNR_{MMSE-DFE} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}$.

  Proof $SNR_{MMSE-DFE} = \dfrac{E_s}{\mathcal{E}_{MMSE-DFE}} = \dfrac{E_s}{\frac{N_0 E_s}{\gamma_0}} = \dfrac{\gamma_0}{N_0} = \dfrac{N_0 e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}}{N_0} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}$.

- Linear zero-forcing equalizer: “zero” ISI completely by channel inversion.
  - $f_{L-ZF}(z) = \dfrac{1}{\rho(z)}$
  - $e(z) = r(z)f(z) - s(z) = \dfrac{r(z)}{\rho(z)} - s(z) = \dfrac{s(z)\rho(z)+w(z)}{\rho(z)} - s(z) = \dfrac{w(z)}{\rho(z)}$.
  - $S_{ee}(z) = E\left[e(z)e^{\#}(z)\right] = \dfrac{S_{ww}(z)}{\rho^2(z)} = \dfrac{N_0\rho(z)}{\rho^2(z)} = \dfrac{N_0}{\rho(z)}$.
  - $\mathcal{E}_{L-ZF} = E\left[\left|e[k]\right|^2\right] = R_{ee}[0] = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} S_{ee}\left(e^{j\omega}\right)d\omega = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \dfrac{N_0}{\rho\left(e^{j\omega}\right)}d\omega$.
  - $SNR_{L-ZF} = \dfrac{E_s}{\mathcal{E}_{L-ZF}} = \dfrac{E_s}{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{N_0}{\rho\left(e^{j\omega}\right)}d\omega} = \left(\dfrac{1}{2\pi}\int_{-\pi}^{\pi}\dfrac{1}{\frac{E_s}{N_0}\rho\left(e^{j\omega}\right)}d\omega\right)^{-1}$.
- $\mathcal{E}_{L-ZF} \ge \mathcal{E}_{L-MMSE} \ge \mathcal{E}_{MMSE-DFE}$ and $SNR_{L-ZF} \le SNR_{L-MMSE} \le SNR_{MMSE-DFE}$.

  Proof $\mathcal{E}_{L-MMSE} = E\left[\left|e[k]\right|^2\right] = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}\dfrac{N_0 E_s}{\rho\left(e^{j\omega}\right)E_s + N_0}d\omega = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}\dfrac{N_0}{\rho\left(e^{j\omega}\right)+\frac{N_0}{E_s}}d\omega$.

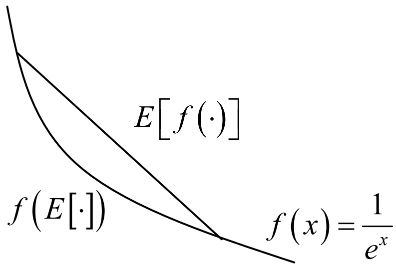
  $\mathcal{E}_{L-ZF} = E\left[\left|e[k]\right|^2\right] = R_{ee}[0] = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} S_{ee}\left(e^{j\omega}\right)d\omega = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}\dfrac{N_0}{\rho\left(e^{j\omega}\right)}d\omega$.

  Because $\dfrac{N_0}{\rho\left(e^{j\omega}\right)} \ge \dfrac{N_0}{\rho\left(e^{j\omega}\right)+\frac{N_0}{E_s}}$,

  we conclude $\mathcal{E}_{L-ZF} = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}\dfrac{N_0}{\rho\left(e^{j\omega}\right)}d\omega \ge \dfrac{1}{2\pi}\int_{-\pi}^{\pi}\dfrac{N_0}{\rho\left(e^{j\omega}\right)+\frac{N_0}{E_s}}d\omega = \mathcal{E}_{L-MMSE}$

Now applying Jensen inequality to $\boldsymbol{\varepsilon}_{MMSE-DFE} = \dfrac{E_s}{e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}}$, we have

$$\boldsymbol{\varepsilon}_{MMSE-DFE} = \frac{E_s}{e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega}} \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{E_s}{e^{\ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)}}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{E_s}{\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1}d\omega = \boldsymbol{\varepsilon}_{L-MMSE}$$



- $\lim_{E_s \to 0} SNR_{MMSE-DFE} = 1$ and $\lim_{E_s \to 0} SNR_{L-ZF} = 0$.

Proof $\lim_{E_s \to 0} SNR_{MMSE-DFE} = \lim_{E_s \to 0} e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho\left(e^{j\omega}\right)\frac{E_s}{N_0}+1\right)d\omega} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln(0+1)d\omega} = 1$

$$\lim_{E_s \to 0} SNR_{L-ZF} = \lim_{E_s \to 0}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\frac{E_s}{N_0}\rho\left(e^{j\omega}\right)}d\omega\right)^{-1} = \lim_{E_s \to 0}\frac{E_s}{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{N_0}{\rho\left(e^{j\omega}\right)}d\omega} = 0$$