Math

- $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = \frac{d}{dx}\left(\int_a^{v(x)} f(t)dt - \int_a^{u(x)} f(t)dt\right) = f(v(x))v'(x) - f(u(x))u'(x)$
- $\int_0^\infty \frac{1}{\sqrt{x}} e^{-\alpha x}dx = \sqrt{\frac{\pi}{\alpha}}$, $\int_0^\infty e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$, $\int_{-\infty}^\infty e^{-(ax^2+bx+c)}dx = \sqrt{\frac{\pi}{\alpha}} e^{\frac{b^2-4ac}{4a}}$
- $F_{X|Y}(x|y) = P[X \le x | Y = y]$, $F_{X,A}(x) = P[\{X \le x\} \cap A]$
- $P[A|X = x] = \lim_{\Delta \to 0} \frac{\int_{x-\Delta}^{x} f_{X,A}(x')dx'}{\int_{x-\Delta}^{x} f_X(x')dx'} = \frac{f_{X,A}(x)}{f_X(x)} = \frac{P[A] f_{X|A}(x)}{f_X(x)}$
- Inequality/bound
  - Markov's: $P[X \ge a] \le \frac{1}{a}EX$, X is a <u>nonnegative</u> r.v.
  - Chebyshev: $P[|X - EX| \ge a] \le \frac{\sigma_X^2}{a^2}$ ☑ One-sided Chebyshev: $P[X \ge EX + a] \le \frac{\sigma_X^2}{\sigma_X^2 + a^2}$,

    $P[X \ge EX - a] \le \frac{\sigma_X^2}{\sigma_X^2 + a^2}$
  - Chernoff: $P[X > a] \le \frac{1}{e^{\lambda a}} E[e^{\lambda X}]$; minimize the RHS w.r.t. λ.

    $E[e^{\lambda X}] = \Phi_X(-i\lambda)$; $P[X < a] \le e^{\lambda a} E[e^{-\lambda X}]$.
  - Union: $P[A \cup B \cup C] \le P[A] + P[B] + P[C]$.
  - Jensen's: For real valued convex function *f*, $f(EX) \le E[f(X)]$.
- **Poisson** $\mathcal{P}(\lambda)$, $e^{-\lambda}\frac{\lambda^i}{i!}$; Ω = N, $0 \le \lambda$, EX = λ, VAR(X) = λ, $\Phi_X(u) = Ee^{iuX} = e^{\lambda(e^{iu}-1)}$ | **Binomial**

  $\binom{n}{k} p^k (1-p)^{n-k}$; np, $np(1-p)$, $(pe^{iu}+1-p)^n$ | **Uniform** $\mathcal{U}(a,b)$, $\frac{a+b}{2}$, $\frac{(b-a)^2}{12}$, $e^{iu\frac{b+a}{2}} \frac{\sin\left(u\frac{b-a}{2}\right)}{u\frac{b-a}{2}}$

  | **Exponential** $\mathcal{E}(\alpha)$, $\frac{1}{\alpha}$, $\frac{1}{\alpha^2}$, $\frac{\alpha}{\alpha - iu}$. | **Laplacian** $\mathcal{L}(\alpha)$, $\frac{\alpha}{2}e^{-\alpha|x|}$;$\alpha > 0$, $\begin{cases} \frac{1}{2}e^{\alpha x} & x < 0 \\ 1 - \frac{1}{2}e^{-\alpha x} & x \ge 0 \end{cases}$, 0, $\frac{2}{\alpha^2}$, $\frac{\alpha^2}{\alpha^2 + u^2}$.
- Majority rule: independent K channels. Each with error probability p.

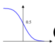
  $P[\mathcal{E}] = P\left[> \frac{K}{2} \text{ error}\right] = \sum_{i=\lceil \frac{K}{2} \rceil}^{K} \binom{K}{i} p^i (1-p)^{K-i} \le 2^K [(1-p)p]^{\frac{K}{2}}$

Gaussian

Real: $\mathcal{N}(m,\sigma^2) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$, $E[e^{-jvX}] = e^{jmv - \frac{1}{2}v^2\sigma^2}$

- $f_{X_1,X_2}(x_1,x_2)=\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}e^{-\frac{\left(\frac{x_1-EX_1}{\sigma_{X_1}}\right)^2-2\rho\left(\frac{x_1-EX_1}{\sigma_{X_1}}\right)\left(\frac{x_2-EX_2}{\sigma_{X_2}}\right)+\left(\frac{x_2-EX_2}{\sigma_{X_2}}\right)^2}{2(1-\rho^2)}}$

  $\rho=\frac{Cov(X_1,X_2)}{\sigma_{X_1}\sigma_{X_2}}=\frac{E(X_1X_2)-EX_1EX_2}{\sigma_{X_1}\sigma_{X_2}}$

- $\mathcal{N}(0,1)$: $Q(z)=\int_z^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$  decreasing function | $Q(0)=\frac{1}{2}$, $Q(-z)=1-Q(z)$ |

  $\boxed{P[X>x]=Q\left(\frac{x-m}{\sigma}\right)}$ | $P[X<x]=1-Q\left(\frac{x-m}{\sigma}\right)=Q\left(-\frac{x-m}{\sigma}\right)$ | $\left(1-\frac{1}{x^2}\right)\frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}}\le Q(x)\le\frac{1}{2}e^{-\frac{x^2}{2}}$

- $\mathcal{N}\left(0,\frac{1}{2}\right)$: $erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx$ | $erfc(z)=1-erf(z)$ | $Q(z)=\frac{1}{2}erfc\left(\frac{z}{\sqrt{2}}\right)=\frac{1}{2}\left(1-erf\left(\frac{z}{\sqrt{2}}\right)\right)$ |

  $erfc(z)=2Q(\sqrt{2}z)$.

Complex: $x,y\sim\mathcal{N}(0,\sigma^2)\rightarrow z=x+iy\sim\mathcal{CN}(0,2\sigma^2)\Rightarrow f_Z(z)=f_{X,Y}(x,y)=\frac{1}{2\pi\sigma^2}e^{-\frac{|z|^2}{2\sigma^2}}=\frac{1}{\pi\sigma_Z^2}e^{-\frac{|z|^2}{\sigma_Z^2}}$

- Independent N-dim $\Rightarrow f_{\vec{Z}}(\vec{z})=\frac{1}{(\pi\sigma^2)^N}e^{-\frac{\|\vec{z}\|^2}{\sigma^2}}$, $\vec{X},\vec{Y}\sim\mathcal{N}\left(0,\frac{\sigma^2}{2}\right)\rightarrow\vec{Z}\sim\mathcal{CN}(0,\sigma^2)$

- Optimal detector: **MAP** (Maximum a posteriori)
  - $\hat{S}_{MAP}=\arg\max_{\vec{s}\in\{\vec{s}_i\}}P[\vec{S}=\vec{s}]f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s})$
  - $\hat{S}_{MAP}=\arg\max_{\{s_i\}}P[\vec{S}=\vec{s}_i|\vec{R}=\vec{r}]$ (a posteriori probability); $\left(\frac{P[\vec{S}=\vec{s}_i]f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s}_i)}{f_{\vec{R}}(\vec{r})}\right)$

- **Optimal decision regions**: $\vec{r}\in\mathcal{R}_i$ iff $P[\vec{S}=\vec{s}_i|\vec{R}=\vec{r}]\ge P[\vec{S}=\vec{s}_j|\vec{R}=\vec{r}]\ \forall j\ne i$, or equivalently, $P[\vec{S}=\vec{s}_i]f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s}_i)\ge P[\vec{S}=\vec{s}_j]f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s}_j)\ \forall j\ne i$.

- ML: $\hat{S}_{ML}=\arg\max_{\vec{s}\in\{\vec{s}_i\}}\underbrace{f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s})}_{\text{Likelihood Function}}$

- $f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s})=f_N(\vec{r}-\vec{s})$ for $\vec{S}$ independent of $\vec{N}$.

- $N\sim\mathcal{N}\left(\vec{0},\frac{N_0}{2}I\right)$, $f_{\vec{N}}(\vec{n})=\frac{1}{(2\pi)^{\frac{M}{2}}\sigma^M}e^{-\frac{1}{2}\frac{\|\vec{n}\|^2}{\sigma^2}}$; or $N\sim\mathcal{CN}(\vec{0},N_0I)$

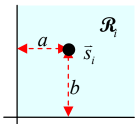
- The **MAP** Receiver for **AWGN** Channel (real or complex N)

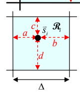
  $\hat{S}_{MAP}=\arg\max_{\vec{s}\in\{\vec{s}_i\}}P[\vec{S}=\vec{s}]f_N(\vec{r}-\vec{s})=\arg\min_{\vec{s}\in\{\vec{s}_i\}}\left(\|\vec{r}-\vec{s}\|^2-N_0\ln P[\vec{S}=\vec{s}]\right)$

  $=\boxed{\arg\max_{\vec{s}\in\{\vec{s}_i\}}\left(\text{Re}\{\vec{s}^H\vec{r}\}+\frac{N_0}{2}\ln P[\vec{S}=\vec{s}]-\frac{\|\vec{s}\|^2}{2}\right)}$

- The **ML** detector for **AWGN** Channel

- $\hat{S}_{ML}\left(\vec{R}=\vec{r}\right)=\arg\min_{\vec{s}\in\{s_i\}}\left(\left\|\vec{r}-\vec{s}\right\|^2\right)=\arg\max_{\vec{s}\in\{s_i\}}\left(\mathrm{Re}\left\{\vec{s}^H\vec{r}\right\}-\frac{\left\|\vec{s}\right\|^2}{2}\right)$





$$P\left[\mathcal{C}\middle|\vec{s}_i\right]=\left(1-Q\left(\frac{a}{\sigma}\right)\right)\left(1-Q\left(\frac{b}{\sigma}\right)\right);\quad P\left[\mathcal{E}\middle|\vec{s}_i\right]=1-P\left[C\middle|\vec{s}_i\right]$$

$$P\left[\mathcal{C}\middle|\vec{s}_i\right]=\left(1-Q\left(\frac{a}{\sigma}\right)-Q\left(\frac{b}{\sigma}\right)\right)\left(1-Q\left(\frac{c}{\sigma}\right)-Q\left(\frac{d}{\sigma}\right)\right)$$

- ML's minimax property: if $P_{ML}\left[\hat{S}\neq\vec{s}_i\middle|\vec{S}=\vec{s}_i\right]$ is independent of *i*, then the ML receiver min the max error prob. for all possible priors $\left\{p_i\triangleq P\left[\vec{s}=\vec{s}_i\right]\right\}$.

- $\left\|\vec{z}_1-\vec{z}_2\right\|^2=\left(\vec{z}_1-\vec{z}_2\right)^H\left(\vec{z}_1-\vec{z}_2\right)=\left\|\vec{z}_1\right\|^2+\left\|\vec{z}_2\right\|^2-2\,\mathrm{Re}\left\{\vec{z}_1^H\vec{z}_2\right\}$
- **Invariance Properties** of Error Probability ☑W.r.t. the translation of signal set ☑ W.r.t. the rotation of the signal points if the noise PDF is invariant when the axes of its arguments are rotated
- **Union bound**: $P\left[\mathcal{E}\middle|\vec{S}=\vec{s}_i\right]=P\left[\vec{R}\notin\mathcal{R}_i\middle|\vec{S}=\vec{s}_i\right]=P\left[\bigcup_{\substack{j\\ j\neq i}}\vec{R}\in\mathcal{R}_j\middle|\vec{S}=\vec{s}_i\right]\leq\sum_{\substack{j\\ j\neq i}}P\left[\vec{R}\in\mathcal{R}_j\middle|\vec{S}=\vec{s}_i\right]$

$\leq P_{BPSK}\left[\vec{S}=\vec{s}_j\middle|\vec{S}=\vec{s}_i\right]=\sum_{\substack{j\\ j\neq i}}Q\left(\frac{\left\|\vec{s}_j-\vec{s}_i\right\|}{2\sigma}\right)$

- $E_b=\frac{E_s}{\log_2 M}$
- **BPSK**: $\hat{S}_{MAP}(r)=\begin{cases}s_1 & r\geq\tau_*\\ s_0 & \text{otherwise}\end{cases}$, $\tau_*=\frac{\sigma^2}{s_1-s_0}\ln\frac{p_0}{p_1}+\frac{s_1+s_0}{2}$

$P[\mathcal{E}]=p_0Q\left(\frac{\tau-s_0}{\sigma}\right)+p_1Q\left(-\frac{\tau-s_1}{\sigma}\right)$ | Binary, ML: $P[\mathcal{E}]=Q\left(\frac{\left|\vec{s}_1-\vec{s}_0\right|}{2\sigma}\right)=Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

| $S\in\left\{s_0=-\sqrt{E_b},s_1=\sqrt{E_b}\right\}$. $E_s=E_b=\frac{\Delta^2}{4}$.

- ***M*-PAM**: $M=2^m$, $s\in\left\{\pm\frac{1}{2}\Delta,\pm\frac{3}{2}\Delta,\pm\frac{5}{2}\Delta,\ldots,\pm\frac{M-1}{2}\Delta\right\}$, $E_s=\Delta^2\frac{M^2-1}{12}$, $P[\mathcal{E}]=\frac{2(M-1)}{M}Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$.

$\frac{\Delta}{\sqrt{2N_0}}=\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}$. $SNR_{norm}=\frac{\log_2 M^2}{M^2-1}\frac{E_b}{N_0}$. $\frac{\Delta}{\sqrt{2N_0}}=\sqrt{3SNR_{norm}}$. | **4-QAM**: $E_s=\frac{\Delta^2}{2}$,

$P[C]=P\left[C\middle|s_0\right]=\left(1-Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)\right)^2$. | ***M*²-QAM**: $E_b\log_2 M^2=E_s=\frac{\left(M^2-1\right)}{2}\Delta^2$.

$P[\mathcal{E}]=1-\left(1-P_{M-PAM}[\mathcal{E}]\right)^2=\frac{4(M-1)}{M}Q-\frac{4(M-1)^2}{M^2}Q^2$. $SNR_{norm}=\frac{\log_2 M^2}{M^2-1}\frac{E_b}{N_0}$.

$\frac{\Delta}{\sqrt{2N_0}}=\sqrt{3\frac{\log_2 M^2}{M^2-1}\frac{E_b}{N_0}}=\sqrt{3SNR_{norm}}$. | ***M*-PSK**: $E_s=\frac{\Delta^2}{4\sin^2\left(\frac{\pi}{M}\right)}$, $Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)\leq P[\mathcal{E}]\leq 2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$.

$$\frac{\Delta}{\sqrt{2N_0}} = \sqrt{2\sin^2\left(\frac{\pi}{M}\right)\log_2 M \frac{E_b}{N_0}} = \sqrt{2(M-1)\sin^2\left(\frac{\pi}{M}\right)SNR_{norm}}.$$ | **M-ary Orthogonal Signals:**

$$\vec{s}_i = \left(0\,0\cdots \underbrace{\sqrt{E_s}}_{i^{th}\text{ position}} \cdots 0\,0\right)^T,\ E_s = \frac{\Delta^2}{2},\ \mathcal{R}_i = \left\{\vec{r}, r_i > r_j, \forall j \neq i\right\},$$

$$p[\mathcal{E}] = p[\mathcal{E}|\vec{s}_1] = 1 - \int\left[Q\left(\frac{-r_1}{\sigma}\right)\right]^{M-1} f(r_1|s_1)dr_1,\ f(r_1|s_1) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\frac{\left(r_1-\sqrt{E_s}\right)^2}{\sigma^2}},\ p[\mathcal{E}] \le (M-1)Q\left(\sqrt{\frac{E_s}{N_0}}\right).$$

$$\frac{\Delta}{\sqrt{2N_0}} = \sqrt{\frac{E_s}{N_0}} = \sqrt{\frac{M}{2}\left(M^{\frac{2}{M}}-1\right)SNR_{norm}}.$$ | **M-dim bi-orthogonal signals**: $\vec{s}_i = \left(0\,0\cdots \underbrace{\pm\sqrt{E_s}}_{i^{th}\text{ position}} \cdots 0\,0\right)^T$,

$$E_s = \frac{\Delta^2}{2},\ p[\mathcal{C}|\vec{s}_1] = \int_0^\infty\left[1-2Q\left(\frac{r_1}{\sigma}\right)\right]^{M-1} f(r_1|s_1)dr_1.$$

- Pairwise error probability: $P\left[\vec{s}_i \to \vec{s}_j\right] = Q\left(\frac{\left\|\vec{s}_i - \vec{s}_j\right\|}{2\sigma_{1D}}\right)$.
- **General cases:** $d_{\min} = \min_{\substack{i,j\\ i\neq j}}\left\|\vec{s}_i - \vec{s}_j\right\|$, $d_i = \min_{\substack{j\\ i\neq j}}\left\|\vec{s}_i - \vec{s}_j\right\|$, $\bar{d} = \frac{1}{M}\sum_i d_i$,

  $Q\left(\frac{\bar{d}}{2\sigma_{1D}}\right), \frac{v}{M}Q\left(\frac{d_{\min}}{2\sigma_{1D}}\right) \le P[\mathcal{E}] \le \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}Q\left(\frac{\left\|\vec{s}_i - \vec{s}_j\right\|}{2\sigma_{1D}}\right) \le (M-1)Q\left(\frac{d_{\min}}{2\sigma_{1D}}\right)$. v = # signals with at least one neighbor at $d_{min} \ge 2$. Small $P[\mathcal{E}]$ if large d_{min}, less number of nearest neighbor.
- $P\left[\vec{s}_i \to \vec{s}_j\right] \triangleq P\left[\hat{S} = \vec{s}_j \middle| \vec{S} = \vec{s}_i\right] \le \iiint\left(f_{\vec{R}|\vec{S}}\left(\vec{x}\middle|\vec{s}_j\right)\right)^{\lambda}\left(f_{\vec{R}|\vec{S}}\left(\vec{x}\middle|\vec{s}_i\right)\right)^{1-\lambda}d\vec{x}$
- **The Bhattacharyya Bound**: $P\left[\vec{s}_i \to \vec{s}_j\right] \le \iiint\sqrt{f_{\vec{R}|\vec{S}}\left(\vec{x}\middle|\vec{s}_j\right)f_{\vec{R}|\vec{S}}\left(\vec{x}\middle|\vec{s}_i\right)}d\vec{x}$.
- Binary **Random Coding**: N-Dim AWGN channel. $\vec{R} = \vec{S} + \vec{N}$, $\vec{R} = \vec{S} + \vec{N} \sim \mathcal{N}(0, \sigma^2 I)$ where $\vec{S}$ is selected with equal probability from two randomly selected symbols $\{\vec{s}_0, \vec{s}_1\}$; $\vec{s}_0, \vec{s}_1 \overset{i.i.d.}{\sim} q_{ND}(\vec{s})$. With

  $q_{ND}(\vec{s}) = \prod_{i=1}^{N} q(s_i)$, $f_{\vec{R}|\vec{S}}(\vec{x}|\vec{s}) = \prod_{i=1}^{N} f(x_i|s_i)$, $P[\vec{s}_0 \to \vec{s}_1] \le \left[\int_x\left[\int_s q(s)\sqrt{f(x|s)}\right]^2\right]^N = 2^{-\gamma(q)N}$.
- Rate R is **achievable** if there exists a sequence of constellations (codes) $\{\vec{s}_1, \ldots, \vec{s}_M\}$, $M = 2^{NR}$ such that $\lim_{N\to\infty} P_{ML}[\mathcal{E}] = 0$. ($M$ symbols = $\log_2 M$ bits per N dimensions. Thus, $R = \frac{\log_2 M}{N}\left[\frac{\text{bits}}{\text{dim}}\right]$)
- Capacity is the maximum achievable rate.
- Capacity Lower Bound of memoryless channel $f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s}) = \prod_{i=1}^{N} f(r_i|s_i)$:

$$R_0 = \max_{q(s)}\left\{-\log_2\int_x\left[\int_s q(s)\sqrt{f(x|s)}ds\right]^2 dx\right\}\left[\frac{\text{bits}}{\text{dim}}\right]$$

- Union bound: $p[\mathcal{E}] = p[\mathcal{E}|\vec{s}_1] \le \sum_{i=2}^{M} P[\vec{s}_1 \to \vec{s}_i]$.

AWGN model. Independent channel. $r_i = s_i + n_i$. $n_i \overset{\text{i.i.d}}{\sim} \mathcal{N}\left(0, \frac{N}{2}\right)$. The transmitter uses some rate-R codebook $\vec{S} \in \left\{\vec{s}_1, \dots, \vec{s}_{2^{NR}}\right\}$ with equally likely codewords (ML).

- Antipodal Antipodal Signaling: $s_i \in \{\pm E_b\}$. $R_0 = -\log_2\left(\frac{1}{2} + \frac{1}{2}e^{-\frac{E_b}{N_0}}\right)$.
- Symmetric M-ary Signaling: $s \in \{s_i\} = \left\{\pm\frac{1}{2}\Delta, \pm\frac{3}{2}\Delta, \pm\frac{5}{2}\Delta, \dots, \pm\frac{M-1}{2}\Delta\right\}$.

  $$R_0(M) = -\log_2 \frac{1}{M^2}\left(\sum_{j=1}^{M}\sum_{i=1}^{M} e^{-(i-j)^2 \frac{3\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right).$$

- BPSK without fading: $P[\mathcal{E}] = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \sim \frac{1}{2}e^{-\frac{E_b}{N_0}}$
- BPSK **Coherent** Detection in Rayleigh Fading (complex)
  - ML, $\|\vec{s}_i\|^2 = E_b$, $\vec{R} = h\vec{S} + \vec{N}$, known $h \sim \mathcal{N}(0, \sigma^2) \Rightarrow f_h(h) = \frac{1}{\pi\sigma_h^2}e^{-\frac{|h|^2}{\sigma_h^2}}$.
  - $\hat{S}_{ML} = \arg\max_{\vec{S}\in\{\vec{s}_0,\vec{s}_1\}} \operatorname{Re}\left\{h^*\vec{s}^H\vec{r}\right\}$ ☑ $P[\mathcal{E}|h] = Q\left(\frac{\|h\vec{s}_1 - h\vec{s}_0\|}{2\sqrt{\frac{N_0}{2}}}\right)$, $P[\mathcal{E}] = E\left[P[\mathcal{E}|h]\right]$
  - $\vec{s}_1 = -\vec{s}_0$, $\|\vec{s}_i\| = E_b \Rightarrow P[\mathcal{E}|h] = Q\left(\sqrt{\frac{2E_b\|h\|^2}{N_0}}\right)$, $SNR = \gamma_0 = \frac{E_b\sigma_h^2}{N_0}$,

    $$P[\mathcal{E}] = \iint Q\left(\sqrt{\frac{2E_b\|h\|^2}{N_0}}\right)\frac{1}{\pi\sigma_h^2}e^{-\frac{|h|^2}{\sigma_h^2}}dh_R dh_I = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}}\right) \approx \frac{1}{4SNR}$$

- BPSK **Noncoherent** Receiver: $\hat{S}_{ML} = \arg\max_{\vec{S}\in\{\vec{s}_0,\vec{s}_1\}} f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s}) = \arg\max_{\vec{S}\in\{\vec{s}_0,\vec{s}_1\}} \left|\vec{r}^H\vec{s}\right|$
  - For $\|\vec{s}_i\| = E_b$, and $\vec{s}_0^H\vec{s}_1 = \vec{s}_1^H\vec{s}_0 = 0$, $P[\mathcal{E}] = \frac{1}{2+SNR}$. $SNR = \frac{E_b\sigma_h^2}{N_0}$.
- ***L*-Diversity Reception**: real: $\vec{r} = \vec{h}s + \vec{n}$, $N \sim \mathcal{N}\left(\vec{0}, \frac{N_0}{2}I\right)$, $\vec{h} \sim \mathcal{N}\left(\vec{0}, \sigma_h^2 I\right)$
  - $\hat{S}_{ML} = \arg\min_{s\in\{s_i\}}\left(\|\vec{r} - \vec{h}s\|^2\right) = \arg\max_{s\in\{s_i\}}\left(\vec{h}^T s\vec{r} - \frac{\|\vec{s}\|^2}{2}\right)$
  - If $s \in \left\{\pm\sqrt{E_b}\right\}$, then $\hat{S}_{ML} = \arg\max_{s\in\{\pm\sqrt{E_b}\}}\left(\vec{h}^T s\vec{r}\right) = \operatorname{sign}\left\{\vec{h}^T\vec{r}\right\}\sqrt{E_b}$.

    Can implement $\vec{h}^T\vec{r}$ by discrete linear combiner (filter). $\vec{r} \to \boxed{\vec{h}} \to \left\langle \vec{r}, \vec{h}\right\rangle$.

    $$P[\mathcal{E}|h] = Q\left(\sqrt{\frac{2E_b\|h\|^2}{N_0}}\right). \text{☑ } P[\mathcal{E}] \le \frac{1}{2(1+SNR)^L}$$

- **Theorem of Reversibility**. If function $T(\vec{r})$ is invertible, then the MAP detector based on $\vec{t} = T(\vec{r})$ minimizes the error probability.
- Given $f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s})$, $T(\vec{R})$ is a **sufficient statistic** for $\vec{s}$ if $f(\vec{r}|\vec{s}, T(\vec{r}))$ is independent of $\vec{s}$. ☑ If $T(\vec{R})$ is a sufficient statistic, then $\hat{S}_{MAP}(T(\vec{R})) = \hat{S}_{MAP}(\vec{R})$.
- **Neyman-Fisher Factorization Theorem**: Given $f(\vec{r}|\vec{s})$, $\vec{t} = T(\vec{r})$ is a sufficient statistic for $\vec{s}$ if and only if $f_{\vec{R}|\vec{S}}(\vec{r}|\vec{s}) = g(T(\vec{r}), \vec{s})h(\vec{r})$.
- Given $f(\vec{r}_1, \vec{r}_2|\vec{s}) = f(\vec{r}_2|\vec{r}_1, \vec{s}) f(\vec{r}_1|\vec{s})$. Disregard $\vec{r}_2$ iff $f(\vec{r}_2|\vec{r}_1, \vec{s}) = f(\vec{r}_2|\vec{r}_1)$.

Waveform

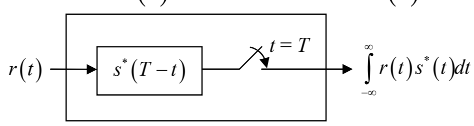
- Parseval Theorem: $\langle x(t), y(t)\rangle = \int x(t) y^*(t)dt = \vec{y}^H \vec{x} = \langle \vec{x}, \vec{y}\rangle$.
- **Correlator receiver**.

$$\hat{S}_{MAP}(t) = \arg \min_{s(t)\in\{s_i(t)\}} \left(\int_{-\infty}^{\infty} |r(t) - s(t)|^2 dt - N_0 \ln P[S(t) = s(t)] \right)$$

$$= \arg \max_{s(t)\in\{s_i(t)\}} \left(\operatorname{Re}\left\{ \int_{-\infty}^{\infty} r(t) s^*(t) dt \right\} + \underbrace{\frac{N_0}{2} \ln P[S(t) = s(t)] - \frac{E_{s(t)}}{2}}_{\gamma_i} \right)$$

$$\hat{S}_{ML}(t) = \arg\min_{s(t)\in\{s_0(t),\dots\}} \|r(t) - s(t)\|^2 = \arg\max_{s(t)\in\{s_0(t),\dots\}} \left(\operatorname{Re}\left\{ \int_{-\infty}^{\infty} r(t) s^*(t) dt \right\} - \frac{\|s(t)\|^2}{2} \right)$$

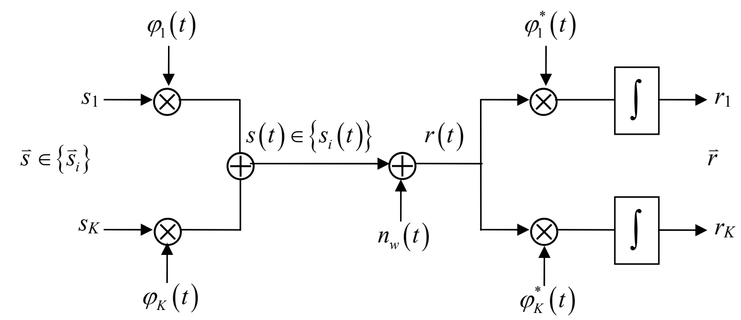
- **Matched Filtering**: Given (complex) $s(t)$, the filter $h(t)$ is **matched** to $s(t)$ at time T if $h(t) = s^*(T - t)$.
- Implement $\left(r(t) * h(t)\right)\Big|_{t=T} = \int_{-\infty}^{\infty} r(t) s^*(t) dt$



- **PPM**: $s(t) \in \{s_0(t), s_1(t), \dots\}$, $s_0(t) = p(t)$, $s_i(t) = p(t - i\Delta)$. $r(t) = s(t) + n_w(t)$. $\Rightarrow$ constant $\|s_i(t)\|^2 = E$.
  - Can implement each $\int_{-\infty}^{\infty} r(t) s_i^*(t) dt$ of optimal detector with
    - $h_i(t) = s_i^*(T - t) = p^*(T - t - i\Delta)$ sampled at time T. or
    - $h(t) = s_0^*(T - t) = p^*(T - t)$ sampled at time $T + i\Delta$.
  - Large T, $\Delta = \frac{T}{2^{RT}}$. Encoding RT bit information into 2^{RT} starting position of the pulse (T sec). Rate $R < \frac{E}{TN_0}\frac{1}{2\ln 2}$ bits/sec. can be achieved reliably.
- **ML Binary signaling:** $s_0(t)$, $s_1(t)$. AWGN $\sim \frac{N_0}{2}$. $P[\mathcal{E}] = Q\left(\frac{\|s_1(t) - s_0(t)\|}{2\sqrt{\frac{N_0}{2}}} \right)$.

- $\{s_1(t),\ldots,s_M(t)\} \Rightarrow K \le M$ orthonormal $\{\varphi_1(t),\ldots,\varphi_K(t)\}$. $s_i(t)=\sum_{j=1}^{K} s_{ij}\varphi_j(t) \Rightarrow \vec{s}_i=[s_{i1}\cdots s_{iK}]^T$.

  $(\vec{s}_i)_j=\int_{-\infty}^{\infty} s_i(t)\varphi_j^*(t)dt$. $n_w(t)$ ~ AWGN with $S(f)=N_0$. $\vec{r}=\vec{s}+\vec{n}$



  $r_i=\int_{-\infty}^{\infty} r(t)\varphi_i^*(t)dt$, $n_i=\int_{-\infty}^{\infty} n_w(t)\varphi_i^*(t)dt$, $\vec{n}\sim\mathcal{CN}(0,N_0I)$.

- ML implementation: Vector (K dim): Get $r_i=\int_{-\infty}^{\infty} r(t)\varphi_i^*(t)dt$, then min $\|\vec{r}-\vec{s}_i\|$. ☑ Continuous (M dim):

  $$\arg\max_{s(t)\in\{s_0(t),\ldots\}}\left(\operatorname{Re}\left\{\int_{-\infty}^{\infty} r(t)s^*(t)dt\right\}-\frac{\|s(t)\|^2}{2}\right).$$

- $\boxed{\int r(t)p(t-\Delta)\,dt=\left(r(t)*p((T-t)-\Delta)\right)\Big|_{t=T}=\left(r(t)*p(T-t)\right)\Big|_{t=T+\Delta}=\left(r(t)*\delta(t+\Delta)*p(T-t)\right)\Big|_{t=T}}$.

- $n_w(t)$ ~ real AWGN with p.s.d. $\frac{N_0}{2}$. $n=[n_w(t)*h(t)]_{t=T}\sim\mathcal{N}\left(0,\frac{N_0}{2}\int_{-\infty}^{\infty} h^2(s)ds\right)$.

- Baseband-Passband: $x^*(t)\overset{\mathcal{F}}{\longleftrightarrow}X^*(-f)$. $\operatorname{Re}\{x(t)\}\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{2}(X(f)+X^*(-f))$. If $x_b(t)$ is a baseband signal, then $\left(\sqrt{2}\left(\sqrt{2}\operatorname{Re}\{e^{j2\pi f_0 t}x_b(t)\}\right)e^{-j2\pi f_0 t}\right)*h_{LP}(t)=x_b(t)$.

- $x_p(t)=\sqrt{2}\operatorname{Re}\{x_b(t)e^{j2\pi f_0 t}\}$. $X_p(f)=\frac{\sqrt{2}}{2}X_b(f-f_0)+\frac{\sqrt{2}}{2}X_b^*(-f-f_0)$.

  $x_p(t)e^{-j2\pi f_0 t}\overset{\mathcal{F}}{\longleftrightarrow}X_p(f+f_0)=\frac{\sqrt{2}}{2}X_b(f)+\frac{\sqrt{2}}{2}X_b^*(-f-2f_0)$. $y_b(t)=\left(\sqrt{2}x_p(t)e^{-j2\pi f_0 t}\right)*h_{LP}(t)$.

  $y_b(t)=x_b(t)$.

- Capacity for Bandlimited AWGN Channels: $C=B\log_2\left(1+\frac{P}{BN_0}\right)$.

- $\langle y[\cdot],x[\cdot]\rangle=\sum_{n=-\infty}^{\infty}x[n]y^*[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$. $\|x[\cdot]\|^2=\sum_{k=-\infty}^{\infty}|x[k]|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2d\omega$.

- F1: $\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{X}(\Omega)e^{j\Omega t}d\Omega = x(t) \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} \hat{X}(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$.

  $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{X}(\omega)e^{jn\omega}d\omega \overset{DTFT}{\rightleftharpoons} \hat{X}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}$. $\hat{X}(\omega+2k\pi) = \hat{X}(\omega)$.

  $\hat{X}(\omega) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T_S}\hat{X}_c\left(\frac{\omega}{T_S}+k\frac{2\pi}{T_S}\right)\right)$. $x[n] = x_c(nT) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\left\{\sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\hat{X}_c\left(\frac{\omega}{T}+k\frac{2\pi}{T}\right)\right)\right\}}_{\hat{X}(\omega)}e^{jn\omega}d\omega$.

- F2: $\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df = x(t) \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$. $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-f)$. $x^*(-t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(f)$.

  $x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}}\hat{X}_{DTFT}(f)e^{jn2\pi f}df \overset{DTFT}{\rightleftharpoons} \hat{X}_{DTFT}(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn2\pi f}$. $x[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}}\underbrace{\frac{1}{T}\sum_{n=-\infty}^{\infty}X\left(\frac{f}{T}+\frac{n}{T}\right)}_{DTFT}e^{j2\pi fk}df$.

  $X_{DTFT}(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty}X\left(\frac{f}{T}+\frac{n}{T}\right)$. $x(t) \overset{\mathcal{F}}{\longrightarrow} g(f)$ iff $g(t) \overset{\mathcal{F}}{\longrightarrow} x(-f)$. $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$.

- $X(f) = \hat{X}(\Omega)\big|_{\Omega=2\pi f}$. $\hat{X}(\Omega) = X(f)\big|_{f=\frac{\Omega}{2\pi}}$. $\hat{X}_{DTFT}(f) = \hat{X}(\omega)\big|_{\omega=2\pi f}$.

- If $x(t)\perp y(t)$, then $X(f)\perp Y(f)$.

- PSD: w.s.s. $X(t) \to \boxed{h(\cdot)} \to$ w.s.s. $Y(t)$. $R_Y(\tau) = R_X(\tau)*h(\tau)*h^*(-\tau)$. $S_Y(f) = S_X(f)|H(f)|^2$.

  $h(\tau)*h^*(-\tau) = \int_{-\infty}^{+\infty} h(\mu)h^*(\mu-\tau)d\mu$. Let $y[k] = y(t)\big|_{t=kT}$ and $R_y[k] = E\left[y[m]y^*[m-k]\right]$, then $R_y[k] = R_y(kT)$.

- **Z-tranform**: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$; $0 \le R_a < |z| < R_b \le \infty$. $x[k] \overset{Z}{\longrightarrow} X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k} \overset{z=e^{j\omega}}{\longrightarrow} X(e^{j\omega})$.

- $x^*[-k] \overset{Z}{\longrightarrow} X^{\#}(z) = \sum_{k=-\infty}^{\infty} x^*[-k]z^{-k} = X^*\left(\frac{1}{z^*}\right) \overset{z=e^{j\omega}}{\longrightarrow} X^*(e^{j\omega})$. $\left(X^{\#}(z)\right)^{\#} = X(z)$.

  $\left(X(z)Y(z)\right)^{\#} = X^{\#}(z)Y^{\#}(z)$. $\left(X(z)+Y(z)\right)^{\#} = X^{\#}(z)+Y^{\#}(z)$. $x[k] = x^*[-k] \Rightarrow X(z) = X^{\#}(z)$.

  $X^{\#}\left(\frac{1}{z_0^*}\right) = X^*(z_0)$. $X^{\#}(z_1) = X^*\left(\frac{1}{z_1^*}\right)$.

- $x[k]*y^*[-k] = \sum_{n=-\infty}^{\infty} x[n]y^*[n-k] \overset{Z}{\longrightarrow} X(z)Y^{\#}(z) = \sum_{k=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} x[n]y^*[n-k]\right)z^{-k}$

- $X(z_0) = 0 \Rightarrow X^{\#}\left(\frac{1}{z_0^*}\right) = 0$. $X^{\#}(z_1) = 0 \Rightarrow X\left(\frac{1}{z_1^*}\right) = 0$. $\left|\lim_{z\to z_0} X(z)\right| = \infty \Rightarrow \left|\lim_{z\to\frac{1}{z_0^*}} X^{\#}(z)\right| = \infty$.

  $\left|\lim_{z\to z_1} X^{\#}(z)\right| = \infty \Rightarrow \left|\lim_{z\to\frac{1}{z_1^*}} X(z)\right| = \infty$. If $X(z) = X^{\#}(z)$, then its poles and zeroes are located symmetrically with respect to the unit circle.

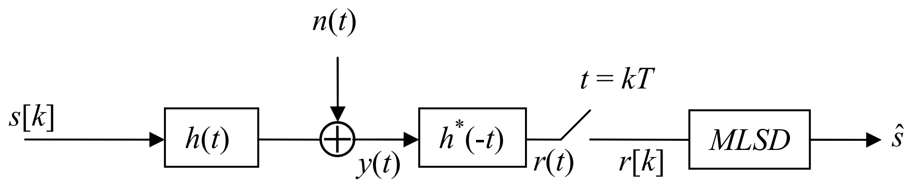
- $H(z)=\frac{B(z)}{A(z)}$ is **minimum phase** $\Leftrightarrow$ all its poles and zeros are inside the unit circles $|z|=1$ $\Leftrightarrow$ both it and its inverse $\frac{1}{H(z)}$ are stable. For a causal minimum phase $h[k]$, the partial energy $\mathcal{E}[n]=\sum_{k=0}^{n}|h[k]|^2$ is maximum for all n among all sequences with the same energy. Thus, minimum-phase signals are maximally concentrated toward time 0 among the space of causal signals for a given magnitude spectrum.
- A polynomial of the form $H(z)=h_0+h_1z^{-1}+\ldots+h_Nz^{-N}=h_0\left(1-z_1z^{-1}\right)\left(1-z_2z^{-1}\right)\cdots\left(1-z_Nz^{-1}\right)$ is minimum phase if all of its roots z_i are inside the unit circle, i.e. $|z_i|<1$.
- **Monic** if $h[0]=1$. Causal $\Rightarrow$ [monic $\Leftrightarrow$ $\lim_{|z|\to\infty}H(z)=1$].
- If $g(z)$ is monic and causal, then $g^{\#}(z)$ is monic and anti-causal. If $g(z)$ is SCAMP, then $g^{-1}(z)$ is also SCAMP. If $g(z)$ is SCAMP, then $g^{-1}(z)$ is monic, anticausal, stable, and maximum phase.
- $\underset{0\le n\le N-1}{x[n]}\xrightarrow{DFT}\underset{0\le k\le N-1}{x_k}=\frac{1}{\sqrt{N}}X\left(e^{j\omega}\right)\Big|_{\omega=k\frac{2\pi}{N}}=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}nk}$

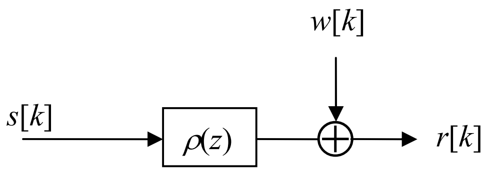
Gap to Capacity Analysis

- Transmit signal from a constellation of M equally likely signals in T [sec]. Information (source) rate = $R=\frac{\log_2 M}{T}\left[\frac{\text{bit}}{\text{sec}}\right]$. Spectral/BW efficiency $\nu=\frac{R}{B}=\frac{\log_2 M}{TB}$. Transmitter (average) power = $P=E\left[\lim_{T\to\infty}\frac{1}{T}\int_T|s(t)|^2\,dt\right]$. $E_s=E\left[\int_{-\infty}^{\infty}|s(t)|^2dt\right]$. $E_b=\frac{PT}{\log_2 M}=\frac{E_s}{\log_2 M}=\frac{P}{R}$. $SNR=\frac{P}{N_0B}=\nu\frac{E_b}{N_0}$. Power efficiency = $\frac{R}{P}=\frac{1}{E_b}$. $SNR\ge SNR_c=2^\nu-1$. Rate normalized SNR: $SNR_{norm}=\frac{SNR}{SNR_c}=\frac{SNR}{2^\nu-1}=\frac{\nu}{2^\nu-1}\frac{E_b}{N_0}>1=0\ dB$. SNR_{norm} = coding gain needed to make $P[\mathcal{E}]\to 0$. To compare modulations, fix $P[\mathcal{E}]$, smaller SNR_{norm} indicates closer to capacity.
- If N = number of real dimension. Then, for PAM, QAM (complex PAM), M-PSK, and M-array orthogonal, N = 1, 2, 2, M, respectively. $2BT=N$.
- Capacity of AWGN channel $C=B\log_2(1+SNR)\ge R\left[\frac{\text{bit}}{\text{sec}}\right]$.
- Transmission of N-Dimensional complex vector $\vec{s}=\left[s[0],\ldots,s[N-1]\right]^T\in\{\vec{s}_1,\ldots,\vec{s}_M\}$.
- For AWGN channel $\vec{r}=\vec{s}+\vec{n}$ where $\vec{n}\sim\mathcal{CN}(0,N_0I)$. If $R=\frac{\log_2 N}{N}<R_0\left[\frac{\text{bit}}{\text{dim}}\right]=\left[\frac{\text{bit}}{\text{channel use}}\right]$, then reliable communication $\left(\lim_{N\to\infty}P[\hat{s}_{ML}\ne\vec{s}]=0\right)$ is possible (using some good codebook.)
- For a channel with BW B, given total time T_m [sec] there are $2BT_m$ real dimensions or $N=BT_m$ complex dimensions. Thus, we can transmit NR [bits] in time T_m [s], which is $\frac{NR}{T_m}=\frac{BT_mR}{T_m}=BR\left[\frac{\text{bit}}{\text{sec}}\right]$.

Single carrier transmission

- For the complex passband channel with BW B, use time-shifted pulses $\{p(t), p(t-T), \ldots, p(t-(N-1)T)\}$.
- If we limit $p(t)$ to BW B, the minimum spacing between two adjacent symbols is $\frac{1}{B}$ [sec].
- **MLSD/MLSE**: Maximum Likelihood Sequence Detection/Estimation





- $h(t) = p(t) * h_b(t)$
- Let $y(t) = \sum_{k=0}^{N-1} s[k]h(t-kT) + n(t)$, where $n(t)$ is complex AWGN with PSD N_0, then $\hat{S}_{ML} = \arg\min_{\vec{s}} -2\,\mathrm{Re}\left\{\sum_{k=0}^{N-1} s^*[k]r[k]\right\} + \left(\sum_{k=0}^{N-1}\sum_{i=0}^{N-1} s[k]s^*[i]\rho[i-k]\right)$, where $r[k] = y(t) * h^*(-t)\big|_{t=kT}$ and $\rho[k] = h(t) * h^*(-t)\big|_{t=kT}$.
- $r[k] = \rho[k] * s[k] + w[k] \xrightarrow{z} r(z) = \rho(z)s(z) + w(z)$. $w[k] = \left(n(t) * h^*(-t)\right)\big|_{kT}$
- $\rho(t) = h(t) * h^*(-t) = \int h(\tau)h^*(\tau - t)d\tau$. $\rho(-t) = \rho^*(t)$. $\rho(0) = \int |h(\tau)|^2 d\tau$ (always real). $\rho(t) \overset{\mathcal{F}}{\rightleftharpoons} Q(f) = |H(f)|^2$.
- $\rho[k] = \rho(kT) = \int h(\tau)h^*(\tau - kT)d\tau$. $\rho[-k] = \rho^*[k]$. $\rho(z) = \sum_{k=-\infty}^{\infty} \rho[k]z^{-k}$. z_0 is a zero of $\rho(z)$ if and only if $\frac{1}{z_0^*} = \frac{1}{|z_0|}\left(\frac{z_0}{|z_0|}\right)$ is a zero of $\rho(z)$. z_0 and $\frac{1}{z_0^*}$ form a pair symmetrical with respect to the unit circle. $\rho^\#(z) = \rho(z)$. $\rho[k] \xrightarrow{DTFT} Q_{DTFT}(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Q\left(\frac{f}{T} + \frac{n}{T}\right) = \sum_{n=-\infty}^{\infty} \rho[n]e^{-jn2\pi f}$.
- If $n(t)$ is white with PSD N_0, then $R_{ww}[k] = N_0\rho[k] \xrightarrow{Z} S_{ww}(z) = N_0\rho(z)$. (Not white unless $\rho[k] = \delta[k]$.)
- The noise sequence $w[k]$ can be generated by a white noise with PSD N_0 and a linear stable causal monic phase filter $\beta(z)$. Because $\rho(z) = \rho^\#(z)$, its poles and zeroes are located symmetrically with respect to the unit circle. Thus, can factorize.

- **The Nyquist Theorem**: $p[k]=p[0]\delta[k]$ iff $\frac{1}{T}\sum_i Q\left(f-\frac{i}{T}\right)=\rho[0]$ and the MLSD is the same as symbol-by-symbol ML detection. $\hat{S}_{ML}=\arg\min_{\vec{s}}\left(\sum_{k=0}^{N-1}\left|s[k]-\frac{r[k]}{\rho[0]}\right|^2\right)$. The min BW (max f – min f) required is $\frac{1}{T}$.

- **Raised cosine pulse** with roll-off factor α: $\rho_\alpha(t)=\frac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}}\frac{\cos\frac{\alpha\pi t}{T}}{1-\frac{4\alpha^2t^2}{T^2}}$. Bandlimited to $\frac{1+\alpha}{2T}$. $Q_\alpha(0)=T$, $Q_\alpha\left(\frac{1}{2T}\right)=\frac{T}{2}$. $\rho[k]=\rho(kT)=\delta[k]$. $Q_\alpha(f)=\frac{T}{2}\left(1+\cos\left(\frac{\pi T}{\alpha}\left(|f|-\frac{1-\alpha}{2T}\right)\right)\right),\frac{1-\alpha}{2T}\le|f|\le\frac{1+\alpha}{2T}$; $T,0\le|f|\le\frac{1-\alpha}{2T};0,|f|\ge\frac{1+\alpha}{2T}$. Decay faster (flatter tail) for higher alpha.

- **Viterbi**: given $\vec{r}=[r_0,r_1,\dots,r_{N-1}]$, $\vec{\rho}=[\rho_0,\rho_1,\dots,\rho_L]$, $\rho_k\ne 0$ only for $-L\le k\le L$, $\mathcal{C}=\{c_1,\dots,c_c\}$. $\hat{s}_{ML}=\arg\min_{\vec{s}\in\mathcal{C}^N}C_{N-1}$ where Cost $C_t=\boxed{-2\operatorname{Re}\left\{\sum_{\ell=0}^{t}s_\ell^*r_\ell\right\}+\left(\sum_{\ell=0}^{t}\sum_{m=0}^{t}s_\ell^*s_m\rho_{\ell-m}\right)}=C_{t-1}+\Delta C(r_t,s_t;\pi_{t-1})$. Define the state at time t by $(s_t,\dots,s_{t-L+1})\Rightarrow\pi_t=\underbrace{(s_t,\dots,s_{t-L+1})}_{L}\in\mathcal{C}^L$. For $t<L$, $\pi_t=(s_t,\dots,s_0)$. $\pi_0=s_0$.

$\boxed{C_0=-2\operatorname{Re}\{s_0^*r_0\}+|s_0|^2\rho_0}$. $\Delta C(r_t,s_t;\pi_{t-1})=-2\operatorname{Re}\{s_t^*r_t\}+2\operatorname{Re}\left\{s_t^*\sum_{k=1}^{\min(t,L)}s_{t-k}\rho_k\right\}+|s_t|^2\rho_0$.

  - For $L=1$, for $t>0$, $\pi_t=s_t$: $C_t=C_{t-1}+\left(-2\operatorname{Re}\{s_t^*r_t\}+2\operatorname{Re}\{s_t^*s_{t-1}\rho_1\}+|s_t|^2\rho_0\right)$.
  - For $L=2$, $\pi_1=(s_1,s_0)$: $C_1=C_0+\Delta C(r_1,s_1;\pi_0)=C_0-2\operatorname{Re}\{s_1^*r_1\}+2\operatorname{Re}\{s_1^*s_0\rho_1\}+|s_1|^2\rho_0$.

    for $t>1$, $\Delta C(r_t,s_t;\pi_{t-1})=-2\operatorname{Re}\{s_t^*r_t\}+2\operatorname{Re}\{s_t^*s_{t-1}\rho_1+s_t^*s_{t-2}\rho_2\}+|s_t|^2\rho_0$.

- SER increases as the length increases.
- $K_1Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)\le P[\mathcal{E}]\le K_2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)+o\left(Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)\right)$ where $d_{\min}$ is the minimum distance between every pair of <u>paths</u>.
- Matched filter bound: single symbol s transmitted over a Baseband channel $h(t)$ modeled by $r(t)=sh(t)+n(t)$ where $n(t)$ is the complex Gaussian noise with PSD N_0.

  $K_1Q\left(\sqrt{\frac{\xi_{\min}^2E_h}{2N_0}}\right)\le P[\mathcal{E}]\le K_2Q\left(\sqrt{\frac{\xi_{\min}^2E_h}{2N_0}}\right)$ where $E_h=\int|h(t)|^2dt$ is the channel energy, and $\xi_{\min}$ is the minimum distance of the symbol constellation.
- Define $\gamma_{MF}=\frac{\xi_{\min}^2E_h}{2N_0}$, $\gamma_{MLSD}=\frac{d_{\min}^2}{2N_0}$ where $d_{\min}$ is the minimum distance of any pair of sequences. $\gamma_{MLSD}\le\gamma_{MF}$. $Q\left(\sqrt{\gamma_{MLSD}}\right)\ge Q\left(\sqrt{\gamma_{MF}}\right)$ because Q is a decreasing function.

- **Correlation and Correlation Spectrum**: Given WSS $x[n], y[n]$,

  $R_{xy}[k] = E\left[x[n]y^*[n-k]\right] \xrightarrow{Z} S_{xy}(z) \overset{?}{=} \hat{E}\left[X(z)Y^\#(z)\right]$ power spectrum density.

  $R_{xy}[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xy}(e^{j\omega})e^{jk\omega}d\omega \overset{DTFT}{\rightleftharpoons} S_{xy}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} R_{xy}[n]e^{-jn\omega}$. $R^*_{yx}[-k] = R_{xy}[k]$. $S^\#_{yx}(z) = S_{xy}(z)$.

  $R_{XX}[-k] = R^*_{XX}[k]$, $S^\#_{XX}(z) = S_{XX}(z)$. $S_{XX}(e^{j\omega})$ is real.

- White process $x[k] \Rightarrow S_{xx}(z) = E_x = E\left[|x[n]|^2\right]$

- $R_{xx}[0] = E\left[|x[k]|^2\right] = \int_{2\pi} \left|S_{xx}(e^{j\omega})\right|^2 d\omega$.

- WSS $x[k] \rightarrow \boxed{h[\cdot]} \rightarrow$ WSS $y[k]$. $R_Y[k] = R_X[k] * h[k] * h^*[-k]$. $S_{yy}(z) = H(z)H^\#(z)S_{xx}(z)$.

  $S_{yy}(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 S_{xx}(e^{j\omega})$.

- For an FIR filter with white input $\left(S_{xx}(z) = E_x = E\left[|x[n]|^2\right]\right)$, $S_{yy}(z) = E_x H(z)H^\#(z)$, and the PSD of the output has zeros symmetrical around the unit circle.

- **Spectral Factorization**: Let $x[k]$ be a stationary process with $S_{xx}(z) = E\left[X(z)X^\#(z)\right]$. If $\int_{-\pi}^{\pi} \ln S_{xx}(e^{j\omega})d\omega > -\infty$, then there exists a $\gamma > 0$ and a unique stable, causal, monic, and minimum phase (SCAMP) filter $g[k] \xrightarrow{Z} G(z)$ such that $S_{xx}(z) = \gamma G(z)G^\#(z)$, $\gamma = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{xx}(e^{j\omega})d\omega}$. We can generate $\tilde{x}[k]$ (same spectrum as x[k]) using a linear filter $G(z)$ with white input.

- Whittle's Construction: If $\ln S_{xx}(z)$ is analytic in $\rho < |z| < \frac{1}{\rho}$ $(\rho < 1)$, then it has the Laurent expansion $\ln S_{xx}(z) = \sum_{k=-\infty}^{\infty} c_k z^{-k}$ where $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{xx}(e^{j\omega})e^{jk\omega}d\omega \xrightarrow{\text{DTFT}} \ln S_{xx}(e^{j\omega})$. c_k is real and even. Then

  $S_{xx}(z) = e^{c_0} e^{\sum_{k=1}^{\infty} c_k z^{-k}} e^{\sum_{k=1}^{\infty} c_{-k} z^{k}}$. $\gamma = e^{c_0} = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{xx}(e^{j\omega})d\omega}$. $G(z) = e^{\sum_{k=1}^{\infty} c_k z^{-k}}$.

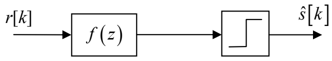
- The Whitening Filter: $S_{xx}(z) = \gamma G(z)G^\#(z)$, $\gamma = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{xx}(e^{j\omega})d\omega}$. If $x[k]$ be the input of a SCAMP filter, then, among the set of SCAMP filters, the whitening filter $\alpha(z) = \frac{1}{G(z)}$ minimizes the output variance. In particular, the output $y[k]$ is a white sequence and $E\left[|y[k]|^2\right] = \gamma$.

  $\gamma = e^{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S_{yy}(e^{j\omega})d\omega} \le \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\ln S_{yy}(e^{j\omega})}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}(e^{j\omega})d\omega = E\left[|y[k]|^2\right]$.

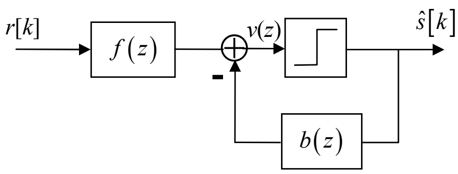
- Optimal Linear Prediction: given $x[t\text{-}1]$, $x[t\text{-}2]$, … $S_{xx}(z) = \gamma\beta(z)\beta^\#(z)$. Find a stable linear predictor $A(z)$ (causal, $a_0 = 0$), $\hat{x}[t] = \sum_{k=1}^{\infty} a_k x[t-k] \xrightarrow{Z} \hat{X}(z) = A(z)X(z)$.

$e[t] = x[t] - \hat{x}[t] \xrightarrow{Z} X(z)(1 - A(z)) = X(z)\alpha(z);\ \alpha(z) = 1 - A(z)$, such that the mean square error $\mathcal{E} = E\left[\left|e[t]\right|^2\right]$ is minimized by a causal monic and stable $\alpha(z)$. $A_{opt}(z) = 1 - \dfrac{1}{\beta(z)}$.

- MMSE Linear Equalizer: scalar: $\hat{s} = fy$. $e = s - fy$. Want to minimize $E\left[|e|^2\right] = E\left[|s - fy|^2\right]$.
  - Completing the squares solution: $\mathcal{E} = E\left[|\hat{s} - s|^2\right] = |f|^2 R_{yy} + R_{ss} - fR_{ys} - f^* R_{sy}$.
    $\left(f - R_{sy}R_{yy}^{-1}\right) R_{yy} \left(f - R_{sy}R_{yy}^{-1}\right)^* = \mathcal{E} - R_{ss} + R_{sy}\left(R_{yy}^{-1}\right)^* R_{ys}$. $f_{MMSE} = R_{sy}R_{yy}^{-1}$. $\mathcal{E}_{MMSE} = R_{ss} - R_{sy}\left(R_{yy}^{-1}\right)^* R_{ys}$.
  - Geometrical solution: $e \perp y \equiv E\left[ey^*\right] = 0$. $f_{MMSE} = R_{sy}R_{yy}^{-1}$.
    $E\left[\mathcal{E}_{MMSE}\right] = E\left[|e|^2\right] = E\left[|s|^2\right] - E\left[|f_{MMSE}y|^2\right] = R_{ss} - R_{sy}R_{yy}^{-1}R_{ys}$. R_{yy} is real.
- **MMSE Linear Equalizer**: $\hat{s}(z) = f(z)y(z)$. $f(z)$: stable IIR, can be non-causal. $e(z) = \hat{s}(z) - s(z)$. Want to minimize $S_{ee}(z) = E\left[e(z)e^{\#}(z)\right]$. $e(z) \perp y(z)$, i.e. $E\left[e(z)y^{\#}(z)\right] = 0$. $f_{opt}(z) = \dfrac{S_{sy}(z)}{S_{yy}(z)}$.
  $S_{ee}(z) = S_{ss}(z) - \dfrac{S_{sy}(z)}{S_{yy}(z)} S_{ys}(z)$.



- **MMSE-DFE**: Causal $b(z)$, $b[0] = 0$ $\left(\lim_{|z|\to\infty} b(z) = 0\right)$. $b(z) = b_1 z^{-1} + b_2 z^{-2} + \cdots$. $\min_{f(z),b(z)} E\left[\left|v[k] - s[k]\right|^2\right]$.
  Assumption: $\hat{s}[k] \approx s[k]$. $v(z) = r(z)f(z) - b(z)\hat{s}(z) \approx r(z)f(z) - b(z)s(z)$.
  $e(z) = v(z) - s(z) = r(z)f(z) - \beta(z)s(z)$. $\beta(z) = b(z) + 1$ is IIR, causal, stable, and monic. $e(z) \perp r(z)$.
  $f(z) = \dfrac{S_{sr}(z)}{S_{rr}(z)}\beta(z)$. $e(z) = \beta(z)\tilde{e}(z)$. $\beta(z)$ from spectral factorization of $S_{\tilde{e}\tilde{e}}(z) = S_{ss}(z) - \dfrac{S_{sy}(z)}{S_{yy}(z)} S_{ys}(z)$ (to minimize $E\left[|e(z)|^2\right]$.)



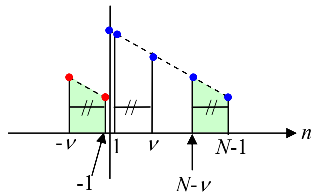
- Discrete-time Equivalent Channel. $S_{ss}(z) = E_s$. $S_{ww}(z) = \rho(z)N_0$. $S_{rr}(z) = \rho(z)\left(\rho(z)S_{ss}(z) + N_0\right)$.
  - Linear MMSE: $f_{MMSE}(z) = \dfrac{S_{ss}(z)}{\rho(z)S_{ss}(z) + N_0} = \dfrac{E_s}{\rho(z)E_s + N_0}$.
    $S_{ee}(z) = \dfrac{N_0 S_{ss}(z)}{\rho(z)S_{ss}(z) + N_0} = \dfrac{N_0 E_s}{\rho(z)E_s + N_0}$. $\mathcal{E}_{L-MMSE} = E\left[\left|e[k]\right|^2\right] = E_s \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \dfrac{1}{\rho\left(e^{j\omega}\right)\dfrac{E_s}{N_0} + 1} d\omega$.

$$SNR_{L-MMSE} = \frac{E_s}{\mathcal{E}_{L-MMSE}}.\quad SNR_{L-MMSE} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\rho\left(e^{j\omega}\right)\frac{E_s}{N_0} + 1} d\omega\right)^{-1}.$$

- MMSE-DFE: If $S_o(z) = \rho(z)E_s + N_0 = \gamma_0 g_0(z) g_0^{\#}(z)$, then $\beta(z) = g_0(z)$. $E\left[\left|e(z)\right|^2\right] = \frac{N_0 E_s}{\gamma_0}$. $\gamma_0 = N_0 e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho(e^{j\omega})\frac{E_s}{N_0}+1\right)d\omega}$. $f(z) = f_{L-MMSE}(z)\beta(z) = \frac{E_s}{\rho(z)E_s + N_0}\beta(z) = \frac{E_s g_0(z)}{\rho(z)E_s + N_0}$. $b_{MMSE-DFE}(z) = \beta(z) - 1 = g_0(z) - 1$. $\mathcal{E}_{MMSE-DFE} = \frac{E_s}{e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho(e^{j\omega})\frac{E_s}{N_0}+1\right)d\omega}}$. $SNR_{MMSE-DFE} = \frac{E_s}{\mathcal{E}_{MMSE-DFE}}$ $= e^{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left(\rho(e^{j\omega})\frac{E_s}{N_0}+1\right)d\omega}$.
- Linear zero-forcing equalizer: $f_{L-ZF}(z) = \frac{1}{\rho(z)}$. $e(z) = r(z)f(z) - s(z) = \frac{w(z)}{\rho(z)}$. $S_{ee}(z) = \frac{N_0}{\rho(z)}$.
  $\mathcal{E}_{L-ZF} = E\left[\left|e[k]\right|^2\right] = R_{ee}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{ee}(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{N_0}{\rho(e^{j\omega})}d\omega$.
  $$SNR_{L-ZF} = \frac{E_s}{\mathcal{E}_{L-ZF}} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\frac{E_s}{N_0}\rho(e^{j\omega})}d\omega\right)^{-1}.$$
- $\mathcal{E}_{L-ZF} \geq \mathcal{E}_{L-MMSE} \geq \mathcal{E}_{MMSE-DFE}$ and $SNR_{L-ZF} \leq SNR_{L-MMSE} \leq SNR_{MMSE-DFE}$.
- $\lim_{E_s \to 0} SNR_{MMSE-DFE} = 1$ and $\lim_{E_s \to 0} SNR_{L-ZF} = 0$.

Multicarrier Transmission

- If $p(t - mT) \perp p(t - nT)$, then $P(t)e^{j2\pi tmT} \perp P(t)e^{j2\pi tnT}$. Basic functions $\left\{P(t)e^{j2\pi t\frac{k}{B}}\right\}$. $k = 0, \ldots, N\text{-}1$.
- $r_k = H_k s_k + w_k$; $r[n] = h[n] \circledast s[n] + w[n]$
- Circular and linear convolution: $h[n] = 0$ for $n < 0$ and $n > \nu$. $s[n] = [s[N-1], \ldots, s[0]]^T$. Cyclic prefix: $\tilde{s}[n] = \begin{cases} s[n], & 0 \leq n \leq N-1 \\ s[n+N], & -\nu \leq n \leq -1 \end{cases}$. Then $(h \circledast s)[n] = (h * \bar{s})[n]$ for $n = 0, \ldots, N\text{-}1$.



- Transmit N samples $s_0, \ldots, s_{N-1}$ at a time. $s_k \xrightarrow{IDFT} s[n]$. Add cyclic prefix $\bar{s} = \left[s[N-\nu], \ldots, s[N-1], s[0], \ldots, s[N-1]\right]$. FIR channel model $x[n] = \sum_{i=0}^{2} h[i]\bar{s}[n-i] + w[n]$ where $w[k] \overset{i.i.d.}{\sim} \mathcal{CN}(0, N_0)$. (Using MATLAB,) have $x[n] = \left[p[N-\nu], \ldots, p[N-1], r[0], \ldots, r[N-1]\right]$. Remove cyclic prefix, get $r[n]$ $\left(\left[r[0], \ldots, r[N-1]\right]\right)$. $r[n] \xrightarrow{DFT} r_k$. $r_k = H_k s_k + w_k$.
- $\sum_{n=0}^{N-1} e^{j2\pi n\frac{k}{N}} = \begin{cases} N\ ;\ \text{if } \frac{k}{N} \in I \\ 0\ ;\ \text{if } \frac{k}{N} \notin I \end{cases}$. $w_k \overset{i.i.d.}{\sim} \mathcal{CN}(0, N_0)$.

- Matrix Formulation: Let $W_N = e^{-j\frac{2\pi}{N}}$. DFT: $\vec{X} = Q\vec{x}$. IDFT: $\vec{x} = Q^H \vec{X}$.

$$\underbrace{\begin{bmatrix} X_{N-1} \\ \vdots \\ X_1 \\ X_0 \end{bmatrix}}_{\vec{X}} = \underbrace{\frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{(N-1)^2} & W_N^{(N-1)(N-2)} & \cdots & 1 \\ \vdots & \vdots & W_N^{(N-j)(N-i)} & \vdots \\ W_N^{(N-1)} & W_N^{(N-2)} & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{N\times N}}_{Q} \underbrace{\begin{bmatrix} x[N-1] \\ \vdots \\ x[1] \\ x[0] \end{bmatrix}}_{\vec{x}}$$

- Q is a unitary matrix, i.e. $Q^H Q = QQ^H = I$. $\left(Q^{-1} = Q^H\right)$. $\vec{x}_2^H \vec{x}_1 = \vec{X}_2^H \vec{X}_1$. $\|\vec{x}\| = \|\vec{X}\|$. $\|\vec{x}_1 - \vec{x}_2\| = \|\vec{X}_1 - \vec{X}_2\|$.

- Circulant matrix

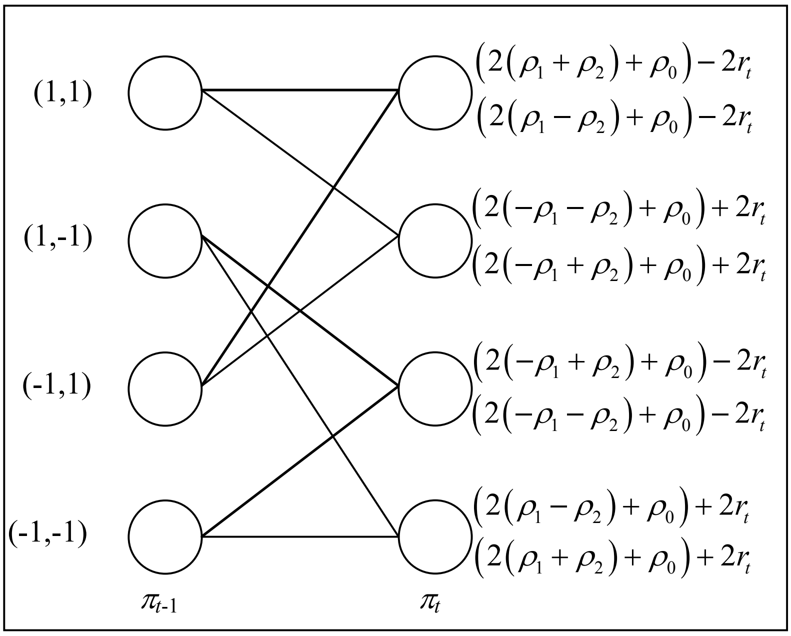
$$H_C = \left[\begin{array}{ccccc|cc} h[0] & h[1] & h[2] & 0 & 0 & & \\ 0 & h[0] & h[1] & h[2] & 0 & & \\ 0 & 0 & h[0] & h[1] & h[2] & & \\ h[2] & 0 & 0 & h[0] & h[1] & h[2] & \\ h[1] & h[2] & 0 & 0 & h[0] & h[1] & h[2] \end{array}\right]$$

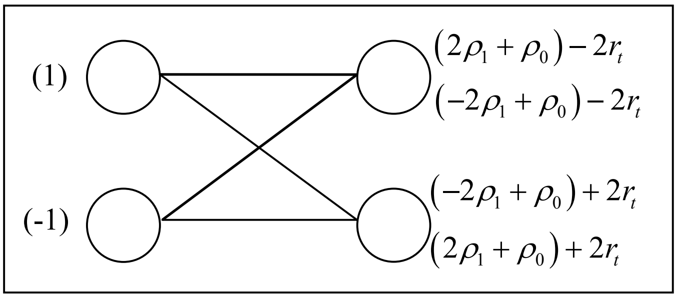
- $QH_C Q^H = \begin{bmatrix} H_{N-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & H_0 \end{bmatrix}$

- $\vec{x} = Q^H \vec{s}$. $\vec{y} = H_C \vec{x} + \vec{w}$. $\vec{r} = Q\vec{y} = QH_C \vec{x} + Q\vec{w} = \begin{bmatrix} H_{N-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & H_0 \end{bmatrix} \vec{s} + \vec{w}'$

- $\hat{s}_{ML} = \arg\min_{\vec{s}} \left\| \vec{r} - \begin{bmatrix} H_{N-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & H_0 \end{bmatrix} \vec{s} \right\|^2 \Leftrightarrow \hat{s}_{ML}[k] = \arg\min_{s} |r_k - H_k s_k|$

- Water filling: $\gamma_k = \frac{|H_k|^2}{\sigma^2}$. $SNR_k = \frac{|H_k|^2 E_k}{\sigma^2} = E_k \gamma_k$. Total power constraint $E = \sum_i E_i$. Maximize rate of reliable transmission by $E_k + \frac{1}{\gamma_k} =$ constant.





(1)
(-1)
$(2\rho_1+\rho_0)-2r_t$
$(-2\rho_1+\rho_0)-2r_t$
$(-2\rho_1+\rho_0)+2r_t$
$(2\rho_1+\rho_0)+2r_t$

Din