# Fourier series

- $\int_0^{T_0} e^{jmw_0 t}\,dt = \begin{cases} T_0 & \text{when } m = 0 \\ 0 & \text{when } m \neq 0 \end{cases}$ ; m is an integer.

$$\int_0^{T_0} e^{jmw_0 t}dt = \frac{1}{jmw_0} e^{jmw_0 t}\Big|_0^{T_0} = \frac{1}{jmw_0}\left(e^{jmw_0 T_0} - 1\right) = \frac{1}{jmw_0}\left(e^{j2pm} - 1\right) = \frac{1}{jmw_0}(1-1) = 0$$

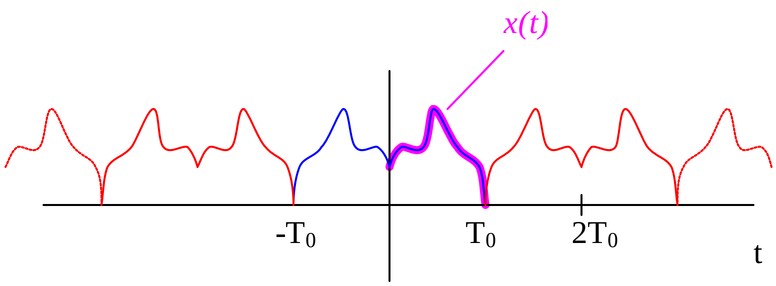
- *r(t)* is **periodic** ↔ $\$(T>0), r(t+T) = r(t), \text{ "}t$
  - **Fundamental period** $T_0$ = smallest T
  - **Fundamental frequency** $w_0 = \dfrac{2p}{T_0}$
- $r_1(t) + r_2(t)$ is periodic ↔ $\dfrac{T_{0_1}}{T_{0_2}}$ is a rational number $\dfrac{k_1}{k_2}$

  Then $T_0 = T_{0_1} k_2 = T_{0_2} k_1$
- $\sum_{k=-M}^{M} c_k e^{jkw_0 t}$ , $\sum_{k=0}^{M} c_k e^{jkw_0 t}$ , $\sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$ has fundamental period $\dfrac{2p}{w_0}$

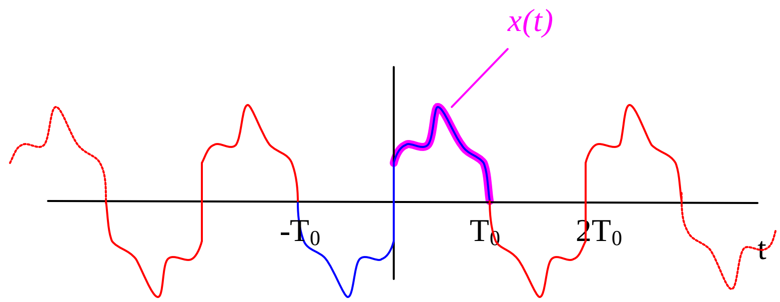
- **Periodic extension** for time-limited *x(t)* { = 0 for *t* < 0 and *t* > $T_0$)

  Let r(t) = $\sum_{n=\infty}^{\infty} x(t - nT_0)$

  - $\tilde{x}(t) = x(t) + x(-t) \Rightarrow \sum_{n=\infty}^{\infty} \tilde{x}(t - n2T_0)$ = **even periodic extension** of x(t)




  - $\tilde{x}(t) = x(t) - x(-t) \Rightarrow \sum_{n=\infty}^{\infty} \tilde{x}(t - n2T_0)$ = **odd periodic extension** of x(t)




- r(t) = $\sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$ ; $w_0 = \frac{2p}{T_0}$
  - $c_0 = \frac{1}{T_0} \int_{T_0} r(t)dt$ = average or DC value of *r(t)*
  - $c_k = \frac{1}{T_0} \int_{T_0} r(t)e^{-jkw_0 t}dt$ = the $k^{th}$ **Fourier coefficient** of *r(t)*
  - $c_k e^{jkw_0 t} + c_{-k}e^{-jkw_0 t}$ = the $k^{th}$ **harmonic component** of *r(t)*
    - $k = 1 \Rightarrow$ fundamental component of *r(t)*
    - *r(t)* is brassier or hasher when it has "heavy" high harmonics
    - $r(t) = c_0 + \sum_{k=1}^{\infty}\left(c_k e^{jkw_0 t} + c_{-k}e^{-jkw_0 t}\right)$

- *r(t)* is real-valued $\left(\overline{r} = r\right) \Rightarrow$
  - $c_{-k} = \overline{c_k}$

$$c_{-k} = \frac{1}{T_0}\int_{T_0} r(t)e^{-j(-k)w_0 t}dt = \overline{\frac{1}{T_0}\int_{T_0} \overline{r(t)}\,\overline{e^{jkw_0 t}}dt} = \overline{\frac{1}{T_0}\int_{T_0} r(t)e^{-jkw_0 t}dt} = \overline{c_k}$$

- $r(t) = c_0 + \sum_{k=1}^{\infty}(a_k \cos(kw_0 t)) + \sum_{k=1}^{\infty}(b_k \sin(kw_0 t))$

  $a_k = 2\,\mathrm{Re}\{c_k\} = \frac{2}{T_0}\int_{T_0} r(t)\cos(kw_0 t)dt$

  $b_k = -2\,\mathrm{Im}\{c_k\} = \frac{2}{T_0}\int_{T_0} r(t)\sin(kw_0 t)dt$

Consider the $k^{th}$ harmonic component of *r(t)*:

$$c_k e^{jkw_0 t} + c_{-k} e^{-jkw_0 t} = c_k e^{jkw_0 t} + \overline{c_k} e^{-jkw_0 t} = c_k e^{jkw_0 t} + \overline{c_k e^{jkw_0 t}} = 2\mathrm{Re}\left\{c_k e^{jkw_0 t}\right\}$$

$$= 2\mathrm{Re}\left\{\left(\mathrm{Re}(c_k) + j\,\mathrm{Im}(c_k)\right)\left(\cos(kw_0 t) + j\sin(kw_0 t)\right)\right\}$$

$$= \underbrace{\left(2\mathrm{Re}(c_k)\right)}_{a_k}\cos(kw_0 t) + \underbrace{\left(-2\mathrm{Im}(c_k)\right)}_{b_k}\sin(kw_0 t)$$

- Example
  - $r(t) = p_{t_0}(t + k2T)$

$$w_0 = \frac{2p}{T_0} = \frac{2p}{2T} = \frac{p}{T}$$

$$c_0 = \frac{2t_0}{2T} = \frac{t_0}{T}$$

$$c_k = \frac{1}{T_0}\int_{T_0} r(t) e^{-jkw_0 t} dt = \frac{1}{2T}\int_{-t_0}^{t_0} e^{-jkw_0 t} dt = \frac{1}{2T}\frac{1}{(-jkw_0)}\left(e^{-jkw_0 t_0} - e^{jkw_0 t_0}\right)$$

$$= \frac{1}{kw_0 T}\frac{1}{(2j)}\left(e^{jkw_0 t_0} - e^{-jkw_0 t_0}\right) = \frac{1}{kw_0 T}\sin(kw_0 t_0) = \frac{1}{kp}\sin\left(kp\frac{t_0}{T}\right)$$

$$a_k = 2\mathrm{Re}\{c_k\} = \frac{2}{kp}\sin\left(kp\frac{t_0}{T}\right)$$

$$b_k = -2\mathrm{Im}\{c_k\} = 0$$

$$r(t) = c_0 + \sum_{k=1}^{\infty}\left(\frac{2}{kp}\sin\left(kp\frac{t_0}{T}\right)\cos(kw_0 t)\right) = \frac{t_0}{T} + \sum_{k=1}^{\infty}\left(\frac{2}{kp}\sin\left(kp\frac{t_0}{T}\right)\cos\left(k\frac{p}{T}t\right)\right)$$

$$\boxed{\sum_{k=0}^{\infty} p_{t_0}(t + 2Tk) = \frac{t_0}{T} + \sum_{k=1}^{\infty}\left(\frac{2}{kp}\sin\left(kp\frac{t_0}{T}\right)\cos\left(k\frac{p}{T}t\right)\right)}$$

  - $r(t) = d(t + k2T)$

$$w_0 = \frac{2p}{T_0} = \frac{2p}{2T} = \frac{p}{T}$$

$$c_0 = \frac{1}{2T}$$

$$c_k = \frac{1}{T_0}\int_{T_0} r(t) e^{-jkw_0 t} dt = \frac{1}{2T}\int_{T_0} d(t) e^{-jkw_0 t} dt = \frac{1}{2T}$$

$$a_k = 2\mathrm{Re}\{c_k\} = \frac{1}{T}$$

$$b_k = -2\mathrm{Im}\{c_k\} = 0$$

$$r(t) = \frac{1}{2T} + \sum_{k=1}^{\infty}\left(\frac{1}{T}\cos(kw_0 t)\right) = \frac{1}{2T} + \sum_{k=1}^{\infty}\left(\frac{1}{T}\cos\left(k\frac{p}{T}t\right)\right)$$

$$\sum_{k=0}^{\infty} d(t + 2Tk) = \frac{1}{2T} + \sum_{k=1}^{\infty}\left(\frac{1}{T}\cos\left(k\frac{p}{T}t\right)\right)$$

- *r(t)* is even { *r(-t) = r(t)* } $\Rightarrow c_{-k} = c_k$

  $$c_{-k} = \frac{1}{T_0}\int_{a}^{a+T_0} r(t)e^{jkw_0 t}dt = -\frac{1}{T_0}\int_{-a}^{-(a+T_0)} r(-t)e^{-jkw_0 t}dt = \frac{1}{T_0}\int_{-(a+T_0)}^{-a} r(t)e^{-jkw_0 t}dt = c_k$$

  Specifically,

  $$c_{-k} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} r(t)e^{-j(-k)w_0 t}dt = -\frac{1}{T_0}\int_{\frac{T_0}{2}}^{-\frac{T_0}{2}} r(-t)e^{-j(k)w_0 t}dt \;\; ; t = -t, dt = -dt$$

  $$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} r(-t)e^{-j(k)w_0 t}dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} r(t)e^{-j(k)w_0 t}dt = c_k$$

- *r(t)* is odd { *r(-t) = -r(t)* } $\Rightarrow c_{-k} = -c_k$

  $$c_{-k} = \frac{1}{T_0}\int_{a}^{a+T_0} r(t)e^{jkw_0 t}dt = -\frac{1}{T_0}\int_{-a}^{-(a+T_0)} r(-t)e^{-jkw_0 t}dt = -\frac{1}{T_0}\int_{-(a+T_0)}^{-a} r(t)e^{-jkw_0 t}dt = -c_k$$

  Specifically,

  $$c_{-k} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} r(t)e^{-j(-k)w_0 t}dt = -\frac{1}{T_0}\int_{\frac{T_0}{2}}^{-\frac{T_0}{2}} r(-t)e^{-j(k)w_0 t}dt \;\; ; t = -t, dt = -dt$$

  $$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} r(-t)e^{-j(k)w_0 t}dt = -\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} r(t)e^{-j(k)w_0 t}dt = -c_k$$

  This proof is similar to the prove that when *r(t)* is even, $c_{-k} = c_k$. The difference is that *r(-t) = -r(t)*, not just *r(t)*, yielding the negative sign in front of the final result.

- *r(t)* is real valued and even $\Rightarrow c_k$'s are all real and $c_{-k} = c_k$

  Real valued *r(t)* $\Rightarrow c_{-k} = \overline{c_k}$

  Even *r(t)* $\Rightarrow c_{-k} = c_k$

  Thus, $c_{-k} = \overline{c_k} = c_k$. Because $\overline{c_k} = c_k$, $c_k$ is real valued.

- *r(t)* is real-valued and odd $\Rightarrow c_k$'s are pure imaginary and $c_{-k} = -c_k$

  Real valued *r(t)* $\Rightarrow c_{-k} = \overline{c_k}$

  Odd *r(t)* $\Rightarrow c_{-k} = -c_k$

Thus, $c_{-k} = \overline{c_k}$ = -$c_k$. Because $\overline{c_k} = -c_k$, $c_k$ is pure imaginary.

- If r(t) is continuous except possibly for some jumps, and has only finitely many jumps in any bounded t-interval,

  then,

  - when t is a part of continuity (non-jump) of r(t), Fourier series converges to r(t)
  - when t is a **jump-point** for r(t), Fourier series converges to the mean value of r(t) across the jump

- **Gibbs Phenomena**

  $S_N(t) = \sum_{k=-N}^{N} c_k e^{jkw_0 t}$

  @ jumps, $S_N$(t)-graph overshoots r(t) graph

  Width of the "overshoot blip" narrow as N → ∞.

  However, height of overshoot doesn't reduce

- **Parseval's Identity**: $\frac{1}{T_0}\int_{T_0} |r(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$

$$\left.\begin{aligned} r(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \\ \overline{r(t)} &= \sum_{m=-\infty}^{\infty} \overline{c_m} e^{-jmw_0 t} \end{aligned}\right\} \Rightarrow |r(t)|^2 = r(t)\overline{r(t)} = \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_k \overline{c_m} e^{j(k-m)w_0 t}$$

$$\int_{T_0} |r(t)|^2 dt = \sum_{k=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty}\left(\int_{T_0} c_k \overline{c_m} e^{j(k-m)w_0 t} dt\right)\right) \sum_{k=-\infty}^{\infty}\left(\sum_{m=-\infty}^{\infty}\left(c_k \overline{c_m}\int_{T_0} e^{j(k-m)w_0 t} dt\right)\right)$$

$$= \sum_{k=-\infty}^{\infty}\left(c_k \overline{c_k}\left(\ldots + 0 + T_0 + 0 + \ldots\right)\right) \quad ; \int_0^{T_0} e^{j(k-m)w_0 t} dt = \begin{cases} T_0 & \text{when k-m = 0} \\ 0 & \text{when k-m} \neq 0 \end{cases}$$

$$= \sum_{k=-\infty}^{\infty}\left(T_0 |c_k|^2\right)$$

- **Fourier series and LTI SISO systems with periodic inputs**

  r(t) = $\sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \xrightarrow{S_{LTI}}$ y(t) $= \sum_{k=-\infty}^{\infty} \hat{H}(kw_0)\cdot c_k \cdot e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} d_k \cdot e^{jkw_0 t}$

  $d_k = \hat{H}(kw_0)\cdot c_k$

  - *y(t)* is also $T_0$-periodic

# Continuous-time Fourier transform (Á, CTFT)

- Nonperiodic signal x(t)

- $\frac{1}{2p}\int_{-\infty}^{\infty}\hat{X}(w)e^{jwt}dw = x(t) \overset{\Im}{\underset{\Im^{-1}}{\rightleftharpoons}} \hat{X}(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$

First, consider time-limited $x(t)$ which $= 0$ for $|t| > \frac{T_0}{2}$

Let $r(t) = \sum_{n=\infty}^{\infty} x(t - nT_0)$.

Fourier series: $r(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$ ; $c_k = \frac{1}{T_0}\int_{T_0} r(t)e^{-jkw_0 t}dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jkw_0 t}dt$

Thus, $r(t) = \sum_{k=-\infty}^{\infty}\left[\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jkw_0 t}dt\right]e^{jkw_0 t}$

$$x(t) = \sum_{k=-\infty}^{\infty}\left(\left[\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jkw_0 t}dt\right]e^{jkw_0 t}\right) \text{ for } t \le \frac{T_0}{2}$$

$$= \frac{1}{2p}\sum_{k=-\infty}^{\infty}\left(\left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jkw_0 t}dt\right]e^{jkw_0 t}w_0\right) \text{ for } t \le \frac{T_0}{2}$$

Let $T_0 \to \infty \Rightarrow \omega_0 \to d\omega,\ k\omega_0 \to \omega,\ \Sigma \to \int$

$$x(t) = \frac{1}{2p}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-jwt}dt\right]e^{jwt}dw = \frac{1}{2p}\int_{-\infty}^{\infty}\hat{X}(w)e^{jwt}dw \quad \forall t$$

- $x(t)$ and $\hat{X}(w)$ are **Á-transform pair** ↔ (one or) both equations ($Á$ , $Á^{-1}$) hold(s).
- In order for $x(t)$ to have a Á-transform, $x(t)$ can't blow up as $t \to \infty$
- In Mathcad, Fourier transform can be found easily by Enter the expression to be transformed. Then, click on the transform variable. Finally, choose Transform / Fourier from the Symbolics menu.

  Mathcad returns a function in the variable ω when perform a Fourier transform. If the expression you are transforming already contains an ω, Mathcad avoids ambiguity by returning a function in the variable ωω instead.
- If

  $x(t) \to 0$ as $|t| \to \infty$ (extremely well-behaved),

  $\frac{d^k}{dt^k}x(t)$ exists for all $k$ { $x(t)$ is infinitely differentiable }, and

$$\frac{d^k}{dt^k}x(t) \to 0 \text{ as } |t| \to \infty,$$

then

- both (Á) and (Á$^{-1}$) hold in strong sense
- $\hat{X}(w) \to 0$ as $|t| \to \infty$

- If $x(t)$ is absolutely integrable,
  then
  - (Á) holds in strong sense
  - $\hat{X}(w)$ is a bounded function of ω
  - (Á$^{-1}$) hold at least in the weak sense

$$x(t) = \frac{1}{2p}\lim_{T\to\infty}\int_{-T}^{T}\hat{X}(w)e^{jwt}dw$$

- If $x(t)$ is square integrable,
  then
  - $\hat{X}(w)$ is also square integrable
  - both (Á) and (Á$^{-1}$) hold in strong sense

- Narrow/sharp in $t \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}$ wide/mushy in $w$

  Wide/mushy in $t \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}$ narrow/sharp in $w$

  Gaussian in $t \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}$ Gaussian in $w$

- Let $x(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{X}(w)$, $x_1(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{X}_1(w)$ and $x_2(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{X}_2(w)$

| $x(t)$ | $\hat{X}(w)$ |
|---|---|
| $\frac{1}{2p}\int_{-\infty}^{\infty}\hat{X}(w)e^{jwt}dw$ | $\int_{-\infty}^{\infty}x(t)e^{-jwt}dt$ |
| $d(t)$ | 1 |

Proof $\int_{-\infty}^{\infty}d(t)e^{-jwt}dt = e^{-jw(0)} = 1$

| | |
|---|---|
| 1 | $2pd(w)$ |

Proof $\frac{1}{\cancel{2p}}\int_{-\infty}^{\infty}\cancel{2p}\,d(w)e^{jwt}dw = 1$

Proof 2 Use duality: $f(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} g(w) \Rightarrow g(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} 2p\,f(-w)$

From $d(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} 1$,

$$1 \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

| | |
|---|---|
| $\alpha$ | $\alpha 2\pi\delta(\omega)$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ |

Proof $\quad \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty}\cancel{2\pi}\,\delta(\omega-\omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$

Proof 2 $\quad$ Use frequency-shift rule: $e^{j\omega_1 t}x(t) \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} \hat{X}(\omega-\omega_1)$

From $1 \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} 2\pi\delta(\omega)$,

$e^{j\omega_0 t}\times 1 \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} 2\pi\delta(\omega-\omega_0)$.

| | |
|---|---|
| • $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ | • $\sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega-k\omega_0)$ |

$$r(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \underset{\mathfrak{I}^{-1}}{\overset{\mathfrak{I}}{\rightleftharpoons}} \hat{R}(\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega-k\omega_0)$$ : discrete spectrum

| | |
|---|---|
| $c_1x_1(t)+c_2x_2(t)$ | $c_1\hat{X}_1(\omega)+c_2\hat{X}_2(\omega)$ |
| $x(t-t_1)$ | $e^{-j\omega t_1}\hat{X}(\omega)$ |

Time-shift rule

Proof $\quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{X}(\omega)e^{j\omega t}d\omega$

$$x(t-t_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{X}(\omega)e^{j\omega(t-t_1)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\hat{X}(\omega)e^{-j\omega t_1}\right]e^{j\omega t}d\omega$$

| | |
|---|---|
| $e^{j\omega_1 t}x(t)$ | $\hat{X}(\omega-\omega_1)$ |

Frequency-shift (or modulation) rule

Proof $\quad \hat{X}(\omega) = \int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$

$$\hat{X}(\omega-\omega_1) = \int_{-\infty}^{\infty}x(t)e^{-j(\omega-\omega_1)t}dt = \int_{-\infty}^{\infty}\left[x(t)e^{j\omega_1 t}\right]e^{-j\omega t}dt$$

| | |
|---|---|
| $\delta(t-t_0)$ | $e^{-j\omega t_0}$ |
| $P_a(t) \quad ; a>0$ | $\frac{2\sin(a\omega)}{\omega} = 2a\,\mathrm{sinc}\left(\frac{a\omega}{\pi}\right)$ |

Proof $\quad \int_{-\infty}^{\infty}P_a(t)e^{-j\omega t}dt = \int_{-a}^{a}e^{-j\omega t}dt = -\frac{1}{j\omega}\left(e^{-j\omega a}-e^{j\omega a}\right) = \frac{2\sin(a\omega)}{\omega}$

| | |
|---|---|
| $\dfrac{\sin(w_0 t)}{p t}$ | $P_{w_0}(w)$ |

Proof $\quad \dfrac{1}{2p}\int_{-\infty}^{\infty} P_{w_0}(w) e^{jwt} dw = \dfrac{1}{2p}\int_{-w_0}^{w_0} e^{jwt} dw = \dfrac{1}{2p\, jt}\left(e^{jtw_0} - e^{-jtw_0}\right)$

$$= \frac{\sin(w_0 t)}{p t}$$

Proof 2 $\quad$ Use duality: $f(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} g(w) \Rightarrow g(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} 2p\, f(-w)$

From $P_a(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \dfrac{2\sin(aw)}{w}$,

$$\frac{2\sin(w_0 t)}{t} \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} 2p\, P_{w_0}(-w) = 2p\, P_{w_0}(w)$$

| | |
|---|---|
| $\dfrac{d}{dt}x(t)$ | $jw\hat{X}(w)$ |

Time-derivative rule

Proof $\quad \dfrac{d}{dt}x(t) = \dfrac{d}{dt}\left(\dfrac{1}{2p}\int_{-\infty}^{\infty}\hat{X}(w)e^{jwt}dw\right) = \dfrac{1}{2p}\int_{-\infty}^{\infty}\left[jw\hat{X}(w)\right]e^{jwt}dw$

| | |
|---|---|
| $-jtx(t)$ | $\dfrac{d}{dw}\hat{X}(w)$ |

Frequency-derivative rule

Proof $\quad \dfrac{d}{dw}\hat{X}(w) = \dfrac{d}{dw}\left(\int_{-\infty}^{\infty}x(t)e^{-jwt}dt\right) = \int_{-\infty}^{\infty}\left[-jtx(t)\right]e^{-jwt}dt$

| | |
|---|---|
| $x(at)$ | $\dfrac{1}{\lvert a\rvert}\hat{X}\left(\dfrac{w}{a}\right)$ |

Time-scaling rule

Proof $\quad$ For *a* > *0*,

$$x(at) = \frac{1}{2p}\int_{-\infty}^{\infty}\hat{X}(w)e^{jwat}dw$$

$$= \frac{1}{2p}\int_{-\infty}^{\infty}\left[\frac{1}{a}\hat{X}\left(\frac{u}{a}\right)\right]e^{jut}du \quad ;u = aw, du = adw$$

For *a* < *0*,

$$x(at) = \frac{1}{2p}\int_{-\infty}^{\infty} \hat{X}(w)e^{jwat}dw$$

$$= \frac{1}{2p}\int_{+\infty}^{-\infty} \left[\frac{1}{a}\hat{X}\left(\frac{u}{a}\right)\right]e^{jut}du \quad ;u = aw, du = adw$$

$$= \frac{1}{2p}\int_{-\infty}^{\infty} \left[\left(-\frac{1}{a}\right)\hat{X}\left(\frac{u}{a}\right)\right]e^{jut}du$$

- For very small $0 < a << 1 \Rightarrow x(at)$ is mushier, more spread out than $x(t) \Rightarrow$ $\frac{1}{a}\hat{X}\left(\frac{u}{a}\right)$ is a taller, narrower version of $\hat{X}(w)$

| | |
|---|---|
| • $x(-t)$ | $\hat{X}(-w)$ |
| $\overline{x(t)}$ | $\overline{\hat{X}(-w)}$ |

Proof $\quad \hat{X}(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$

$$\hat{X}(-w) = \int_{-\infty}^{\infty} x(t)e^{jwt}dt$$

$$\overline{\hat{X}(-w)} = \overline{\int_{-\infty}^{\infty} x(t)e^{jwt}dt} = \int_{-\infty}^{\infty}\left[\overline{x(t)}\right]e^{-jwt}dt$$

| | |
|---|---|
| $x_1(t) * x_2(t)$ | $\hat{X}_1(w)\cdot\hat{X}_2(w)$ |

Convolution-in-time Rule

Proof $\quad$ Let $y(t) = \int_{-\infty}^{\infty} x_1(t)x_2(t-t)dt$

$$\hat{Y}(w) = \int_{-\infty}^{\infty} y(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(t)x_2(t-t)dt\right]e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} x_1(t)\left[\int_{-\infty}^{\infty} x_2(t-t)e^{-jwt}dt\right]dt$$

$$= \int_{-\infty}^{\infty} x_1(t)\left[e^{-jwt}\hat{X}_2(w)\right]dt \quad \text{;time-shift rule}$$

$$= \hat{X}_2(w)\int_{-\infty}^{\infty} x_1(t)e^{-jwt}dt = \hat{X}_1(w)\cdot\hat{X}_2(w)$$

| | |
|---|---|
| $x_1(t)\cdot x_2(t)$ | $\frac{1}{2p}\hat{X}_1(w) * \hat{X}_2(w)$ |

Convolution-in-frequency Rule

Proof Let $\hat{Y}(w)=\int_{-\infty}^{\infty}\hat{X}_1(w')\hat{X}_2(w-w')dw'$ ???

$$y(t)=\frac{1}{2p}\int_{-\infty}^{\infty}\hat{Y}(w)e^{jwt}dw$$

$$=\frac{1}{2p}\int_{-\infty}^{\infty}\left[\frac{1}{2p}\int_{-\infty}^{\infty}\hat{X}_1(w')\hat{X}_2(w-w')dw'\right]e^{jwt}dw$$

$$=\frac{1}{2p}\int_{-\infty}^{\infty}\hat{X}_1(w')\left[\frac{1}{2p}\int_{-\infty}^{\infty}e^{jwt}\hat{X}_2(w-w')dw\right]dw'$$

$$=\frac{1}{2p}\int_{-\infty}^{\infty}\hat{X}_1(w')\left[e^{jw't}x_2(t)\right]dw' \text{ ;frequency-shift rule}$$

$$=x_2(t)\int_{-\infty}^{\infty}\frac{1}{2p}\hat{X}_1(w')e^{jw't}dw'=x_1(t)\cdot x_2(t)$$

Proof 2 Use duality: $f(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}g(w)\Rightarrow g(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}2p\,f(-w)$

From $x_1(t)*x_2(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\hat{X}_1(w)\cdot\hat{X}_2(w)$,

$x_1(t)\cdot x_2(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}2p\left(\hat{X}_1*\hat{X}_2\right)(-w)$ ???

| $\overline{x(t)}$ | $\overline{\hat{X}(-w)}$ |
|---|---|
| $e^{-at}u(t)$ | $\frac{1}{a+jw}$ |

$$\int_{-\infty}^{\infty}\left[e^{-at}u(t)\right]e^{-jwt}dt=\int_{0}^{\infty}e^{-(a+jw)t}dt=\frac{1}{a+jw}$$

| | |
|---|---|
| $e^{at}u(-t)$ | $\frac{1}{a-jw}$ |

$$\int_{-\infty}^{\infty}\left[e^{at}u(-t)\right]e^{-jwt}dt=\int_{-\infty}^{0}e^{(a-jw)t}dt=\frac{1}{a-jw}$$

Or use $x(-t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\hat{X}(-w)$,

know that $e^{-at}u(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\frac{1}{a+jw}$, then $e^{-a(-t)}u(-t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\frac{1}{a-jw}$.

| | |
|---|---|
| $e^{-a\lvert t\rvert}$ | $\frac{-2jw}{a^2+w^2}$ |

$$e^{-a|t|}=e^{-at}u(t)-e^{at}u(-t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\frac{1}{a+jw}-\frac{1}{a-jw}=\frac{-2jw}{a^2+w^2}$$

| | |
|---|---|
| $u(t)$ | $\frac{1}{jw}+pd(w)$ |

Define $\operatorname{sgn}(t)=\begin{Bmatrix}1 & t>0\\ 0 & t=0\\ -1 & t<0\end{Bmatrix}=\lim_{a\to 0^+}\left[e^{-at}u(t)-e^{at}u(-t)\right]$

$$\operatorname{sgn}(t)=\lim_{a\to 0^+}\left[e^{-at}u(t)-e^{at}u(-t)\right]\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\lim_{a\to 0^+}\frac{-2jw}{a^2+w^2}=\frac{-2jw}{w^2}=\frac{-2j}{w}$$

$$u(t)=\frac{1}{2}(\operatorname{sgn}(t)+1)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\frac{1}{2}\left(\frac{-2j}{w}+2pd(w)\right)=\frac{1}{jw}+pd(w)$$

- Note $\frac{d}{dt}u(t)=d(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}jw\left(\frac{1}{jw}+pd(w)\right)=1$

| | |
|---|---|
| $\cos(w_c t)$ | $pd(w-w_c)+pd(w+w_c)$ |

Proof Use $\cos(w_c t)=\frac{1}{2}e^{jw_c t}+\frac{1}{2}e^{-jw_c t}$

Recall that $e^{jw_0 t}\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}2pd(w-w_0)$

| | |
|---|---|
| $\sin(w_0 t)$ | $\frac{p}{j}\left(d(w-w_0)-d(w+w_0)\right)$ |
| $x(t)\times\cos(w_c t)$ | $\frac{1}{2}\hat{X}(w-w_c)+\frac{1}{2}\hat{X}(w+w_c)$ |

Proof $x(t)\times\cos(w_c t)=\frac{1}{2}x(t)e^{jw_c t}+\frac{1}{2}x(t)e^{-jw_c t}$

Then, use Frequency-shift/modulation rule:

$e^{jw_1 t}x(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\hat{X}(w-w_1)$.

| | |
|---|---|
| $ke^{-at^2}$ | $\left(k\sqrt{\frac{p}{\lvert a\rvert}}\right)e^{-\left(\frac{1}{4\lvert a\rvert}\right)w^2}$ |

Thus, Gaussian $\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}$ Gaussian.

- **Duality**: $f(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}g(w)\Rightarrow g(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}2p\,f(-w)$

$$g(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

$$g(p) = \int_{-\infty}^{\infty} f(z)\, e^{-jpz}\, dz$$

$$g(t) = \int_{-\infty}^{\infty} f(-w)\, e^{jwt}\, dw \quad ; z = -w, \;\; p = t$$

$$= \frac{1}{2p} \int_{-\infty}^{\infty} \left[ 2p\, f(-w) \right] e^{jwt}\, dw$$

- **Parseval's Identity**: $\int_{-\infty}^{\infty} \left| x(t) \right|^2 dt = \frac{1}{2p} \int_{-\infty}^{\infty} \left| \hat{X}(w) \right|^2 dw$ ,if *x(t)* is square integrable

$$x(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{X}(w)$$

$$\overline{x(t)} \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \overline{\hat{X}(-w)}$$

$$y(t) = x(t) \cdot \overline{x(t)} = \left| x(t) \right|^2 \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \frac{1}{2p} \hat{X}(w) * \overline{\hat{X}(-w)} = \hat{Y}(w)$$

By definition $\hat{Y}(w) = \int_{-\infty}^{\infty} y(t)\, e^{-jwt}\, dt$

Thus, $\int_{-\infty}^{\infty} \left| x(t) \right|^2 e^{-jwt}\, dt = \frac{1}{2p} \hat{X}(w) * \overline{\hat{X}(-w)} \overset{?}{=} \frac{1}{2p} \int_{-\infty}^{\infty} \hat{X}(m) * \overline{\hat{X}(-(w-m))}\, dm$

$$@w = 0 \Rightarrow \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt = \frac{1}{2p} \int_{-\infty}^{\infty} \hat{X}(m) * \overline{\hat{X}(m)}\, dm = \frac{1}{2p} \int_{-\infty}^{\infty} \left| \hat{X}(m) \right|^2 dm$$

- *x(t)* is real-valued $\left( x(t) = \overline{x(t)} \right) \Rightarrow \hat{X}(-w) = \overline{\hat{X}(w)}$

$$\hat{X}(-w) = \int_{-\infty}^{\infty} x(t)\, e^{jwt}\, dt = \int_{-\infty}^{\infty} \overline{x(t)}\, e^{jwt}\, dt = \overline{\int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt} = \overline{\hat{X}(w)}$$

*x(t)* is even $\left( x(t) = x(-t) \right) \Rightarrow \hat{X}(w)$ is also even $\Rightarrow \hat{X}(-w) = \hat{X}(w)$

$$\hat{X}(-w) = \int_{-\infty}^{\infty} x(t)\, e^{-j(-w)t}\, dt = \int_{-\infty}^{\infty} x(-t)\, e^{-jwt}\, dt \;\; ; t = -t$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt = \hat{X}(w)$$

*x(t)* is odd $\left( x(t) = -x(-t) \right) \Rightarrow \hat{X}(w)$ is also odd $\Rightarrow \hat{X}(-w) = -\hat{X}(w)$

$$\hat{X}(-w) = \int_{-\infty}^{\infty} x(t)\, e^{-j(-w)t} dt = \int_{-\infty}^{\infty} x(-t)\, e^{-jwt}\, dt \;\; ; t = -t$$

$$= \int_{-\infty}^{\infty} -x(t)\, e^{-jwt}\, dt = -\hat{X}(w)$$

| |
|---|
| *x(t)* is real and even $\rightarrow$ so is $\hat{X}(w)$ |

$$\left.\begin{aligned} \hat{X}(-w) &= \overline{\hat{X}(w)} \\ \hat{X}(-w) &= \hat{X}(w) \end{aligned}\right\} \Rightarrow \overline{\hat{X}(w)} = \hat{X}(w) \Rightarrow \hat{X}(w) \text{ is real}$$

| |
|---|
| *x(t)* is real and odd $\rightarrow$ $\hat{X}(w)$ is pure imaginary and odd |

$$\left.\begin{aligned} \hat{X}(-w) &= \overline{\hat{X}(w)} \\ \hat{X}(-w) &= -\hat{X}(w) \end{aligned}\right\} \Rightarrow \overline{\hat{X}(w)} = -\hat{X}(w) \Rightarrow \hat{X}(w) \text{ is pure imaginary}$$