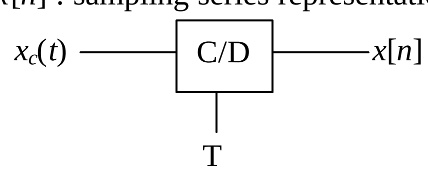
# Sampling & Reconstruction of continuous-time signals

- $x[n] = x_c(nT_S) = x_R(nT_R);\ -\infty < n < \infty$
- $\hat{X}(w) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T_S}\hat{X}_c\left(\frac{w}{T_S} + k\frac{2p}{T_S}\right)\right)\ \forall w, T_S$
- $\hat{X}(w) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T_R}\hat{X}_R\left(\frac{w}{T_R} + k\frac{2p}{T_R}\right)\right)\ \forall w, T_R = \frac{1}{T_R}\hat{X}_R\left(\frac{w}{T_R}\right)$ $-p \le w \le p$; repeat
- $x_R(t) = \frac{1}{2p}\int_{-\frac{p}{T_R}}^{\frac{p}{T_R}} T_R \hat{X}(\Omega T_R) e^{j\Omega t} d\Omega\ ;\forall\, t = \frac{1}{2p}\int_{-p}^{p}\hat{X}(m) e^{j\frac{m}{T_R}t} dm$

  $$= \sum_{n=-\infty}^{\infty} x[n]\frac{\sin\frac{p}{T}(t - nT)}{\frac{p}{T}(t - nT)}\ ;\forall t$$
- $\hat{X}_R(\Omega) = T\hat{X}(w = \Omega T)\, p_{\frac{p}{T}}(\Omega) = T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=\Omega T}$
- If $T_R = T_S = T$ , $\Omega_S = \frac{2p}{T_S} > 2\Omega_m$ or $T_S < \frac{p}{\Omega_m}$
  - $\hat{X}(w) = \frac{1}{T}\hat{X}_c\left(\frac{w}{T}\right)$ ; $-p \le w \le p$; repeat
  - $\hat{X}_c(\Omega) = \hat{X}_R(\Omega) = T\hat{X}(w = \Omega T)\, p_{\frac{p}{T}}(\Omega) = T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=\Omega T}$
  - $x_c(t) = x_R(t)$

- $x[n] = x_c(nT)\ ;\ -\infty < n < \infty$
  - $x[n]$ : sampling series representation for $x_c(t)$




- **Deconstruction equation**: $\hat{X}(w) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\hat{X}_c\left(\frac{w}{T} + k\frac{2p}{T}\right)\right)\ \forall w$,T (**D**)
  - $\Rightarrow$ sum of scaled, shifted replicas of $\hat{X}_c(\Omega)$

- $$\frac{1}{T}\hat{X}_c\left(\frac{w}{T}\right) \Rightarrow \Omega = \frac{w}{T} \Rightarrow$$ what happens at $\Omega = \Omega_0$, happens at $w = \Omega_0 T$

- Space between centers of replicas $\Rightarrow \Delta\Omega = \dfrac{2p}{T} \Rightarrow \Delta w = \dfrac{2p}{T}T = 2p$

- In general, replicas "collide" in $w$-space

Proof $$x_c(t) = \frac{1}{2p}\int_{-\infty}^{\infty}\hat{X}(\Omega)e^{j\Omega t}d\Omega$$

Sector the integration:

$$x_c(t) = \frac{1}{2p}\sum_{k=-\infty}^{\infty}\left(\int_{k\frac{2p}{T}-\frac{p}{T}}^{k\frac{2p}{T}+\frac{p}{T}}\hat{X}_c(\Omega)e^{j\Omega t}d\Omega\right)$$

Then, let $m = \Omega - k\dfrac{2p}{T} \Rightarrow dm = d\Omega$

$$x_c(t) = \frac{1}{2p}\sum_{k=-\infty}^{\infty}\left(\int_{-\frac{p}{T}}^{\frac{p}{T}}\hat{X}_c\left(m + k\frac{2p}{T}\right)e^{j\left(m+k\frac{2p}{T}\right)t}dm\right)$$

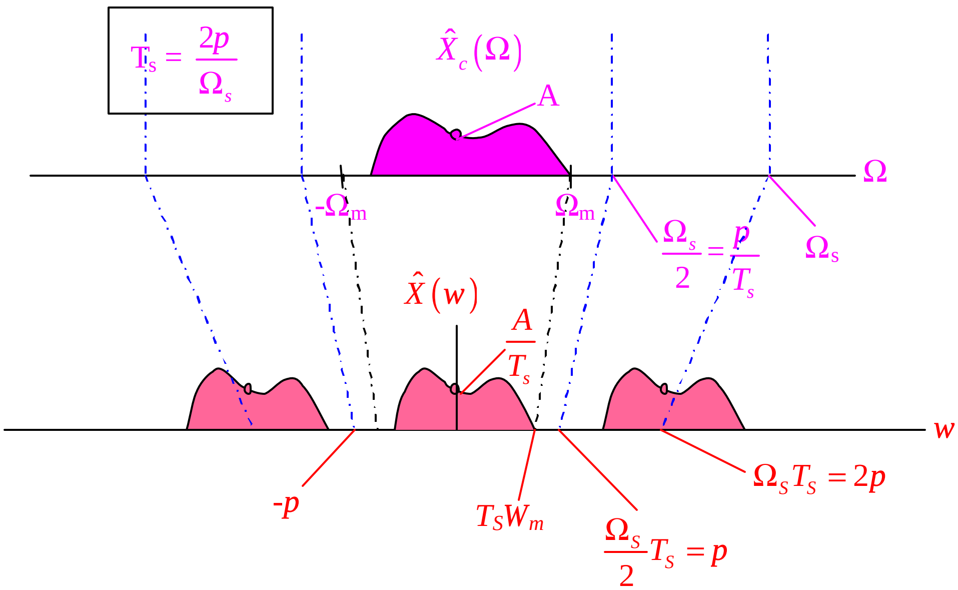
Let $w = mT \Rightarrow dw = Tdm$

$$x_c(t) = \frac{1}{2p}\sum_{k=-\infty}^{\infty}\left(\int_{-p}^{p}\frac{1}{T}\hat{X}_c\left(\frac{w}{T} + k\frac{2p}{T}\right)e^{j\left(\frac{w}{T}+k\frac{2p}{T}\right)t}dw\right)$$

$$x_c(nT) = x[n] = \frac{1}{2p}\sum_{k=-\infty}^{\infty}\left(\int_{-p}^{p}\frac{1}{T}\hat{X}_c\left(\frac{w}{T} + k\frac{2p}{T}\right)e^{j\left(\frac{w}{T}+k\frac{2p}{T}\right)nT}dw\right)$$

$$= \frac{1}{2p}\sum_{k=-\infty}^{\infty}\left(\int_{-p}^{p}\frac{1}{T}\hat{X}_c\left(\frac{w}{T} + k\frac{2p}{T}\right)e^{jnw}\cancelto{1}{e^{jkn2p}}dw\right)$$

$$= \frac{1}{2p}\int_{-p}^{p}\underbrace{\left\{\sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\hat{X}_c\left(\frac{w}{T} + k\frac{2p}{T}\right)\right)\right\}}_{\hat{X}(w)}e^{jnw}dw$$




## Shannon-Nyquist Sampling Theorem

- From picture, for no aliasing ⇒ need $T_s W_m < p$

- If $x(t)$ is $W_m$-bandlimited,
  can recover $x(t)$ exactly from the discrete sequence of samples provided that
  $\Omega_S = \frac{2p}{T_S} > 2\Omega_m$ or $T_S < \frac{p}{\Omega_m}$ = **Nyquist interval** for $x_c(t)$

  - $\Omega_m$ : bandwidth of $x_c(t)$

- Given $x_c(t)$ with
  - $\hat{X}_c\left(|\Omega| \geq \frac{p}{T}\right) = 0$
  - $T < \frac{p}{\Omega_m}$ ; $\Omega_m$ : bandwidth of $x_c(t)$

  $x_c(t)$ is determined completely by $x[n] = x_c(nT)$ , ∀n

  - $\hat{X}(w) = \frac{1}{T}\hat{X}_c\left(\frac{w}{T}\right)$ ; $-p \leq w \leq p$

    $\hat{X}_c(\Omega) = T\hat{X}(\Omega T)$ ; $-\frac{p}{T} \leq w \leq \frac{p}{T}$

  ⇒ no overlap

- $x_c(t) = \frac{1}{2p}\int_{-\infty}^{\infty} \hat{X}_c(\Omega) e^{j\Omega t} d\Omega = \frac{1}{2p}\int_{-\frac{p}{T}}^{\frac{p}{T}} T\hat{X}(\Omega T) e^{j\Omega t} d\Omega$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{p}{T}(t-nT)}{\frac{p}{T}(t-nT)} \ ; \forall t$$

Proof

$$x_c(t) = \frac{1}{2p}\int_{-\infty}^{\infty} \hat{X}_c(\Omega) e^{j\Omega t} d\Omega = \frac{1}{2p}\int_{-\frac{p}{T}}^{\frac{p}{T}} T\hat{X}(\Omega T) e^{j\Omega t} d\Omega$$

$$= \frac{1}{2p}\int_{-\frac{p}{T}}^{\frac{p}{T}} T\left(\sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega T}\right) e^{j\Omega t} d\Omega$$

$$= \sum_{n=-\infty}^{\infty} \left\{ Tx[n] \cdot \left( \frac{1}{2p}\int_{-\frac{p}{T}}^{\frac{p}{T}} e^{j\Omega(t-nT)} d\Omega \right) \right\} = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{p}{T}(t-nT)}{\frac{p}{T}(t-nT)} \ ; \forall t$$

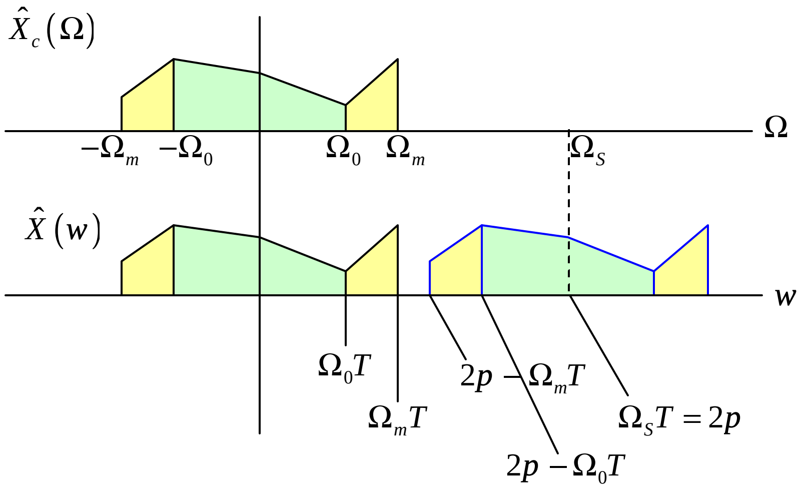
- $z_{practical}(t) \xrightarrow{\hat{H}(\Omega) = p_{\Omega_m}(\Omega)} c_0 x(t) = \frac{2a}{T_0} x(t)$
- $z_{ideal}(t) = \xrightarrow{\hat{H}(\Omega) = T_0 \cdot p_{\Omega_m}(\Omega)} x(t)$

- To find maximum $T_S$ for signal that has high-frequency don't-care region

  $\Rightarrow$ Interested in $\hat{X}_c(\Omega)$ when $|\Omega| \le \Omega_0$, Don't care what happens when $\Omega_0 < |\Omega| \le \Omega_m$

  And $\hat{X}_c(\Omega) = 0$ when $|\Omega| \ge \Omega_m$.

  - From $\hat{X}(w) = \sum_{k=-\infty}^{\infty} \left( \frac{1}{T_S} \hat{X}_c\left( \frac{w}{T_S} + k\frac{2p}{T_S} \right) \right)$, we can see that the interesting region (in green) of $\hat{X}_c\left(\frac{w}{T}\right)$ ends at $\Omega_0 T$, and $\hat{X}_c\left(\frac{w}{T} - \frac{2p}{T}\right)$ start at $2p - \Omega_m T$.




Note that we can allow the don't-care region (in yellow) to overlap, so only need $2p - \Omega_m T > \Omega_0 T$ so that the don't care region of $\hat{X}_c\left(\frac{w}{T} - \frac{2p}{T}\right)$ will not overlap the interesting region of $\hat{X}_c\left(\frac{w}{T}\right) \Rightarrow 2p > \Omega_0 T + \Omega_m T \;\Rightarrow T < \frac{2p}{\Omega_0 + \Omega_m}$

also need $2p - \Omega_0 T > \Omega_m T$ so that the don't care region of $\hat{X}_c\left(\frac{w}{T}\right)$ will not overlap the interesting region of $\hat{X}_c\left(\frac{w}{T} - \frac{2p}{T}\right) \Rightarrow T < \frac{2p}{\Omega_0 + \Omega_m}$.

In this case, since the regions of $\hat{X}_c(\Omega)$ are symmetric, both requirements yield the same result: $T < \frac{2p}{\Omega_0 + \Omega_m}$.

Usually, we need $T < \frac{p}{\Omega_m}$, here we can have $T$ larger: as large as $\frac{2p}{\Omega_0 + \Omega_m}$.

$$\Omega_m > \Omega_0 \Rightarrow \frac{2p}{\Omega_0 + \Omega_m} > \frac{p}{\Omega_m}$$

- $$x[n] \rightarrow \boxed{D/C}_{\uparrow T} \xrightarrow{x_R(t)} \boxed{p_{\Omega_0}(\Omega)} \rightarrow y_c(t)$$

  $y_c(t) = x_c(t)$ only in the frequency region of interest

  $\hat{X}_R(\Omega) = \hat{X}_c(\Omega)$ for $|\Omega| \le \Omega_0$ , junk for $\Omega_0 < |\Omega| \le \Omega_m$ , 0 for $|\Omega| > \Omega_m$

  $\hat{Y}_c(\Omega) = \hat{X}_R(\Omega)\, p_{\Omega_0}(\Omega) = \hat{X}_c(\Omega)\, p_{\Omega_0}(\Omega) \;\Rightarrow$ no junk

## Reconstruction of continuous-time signals

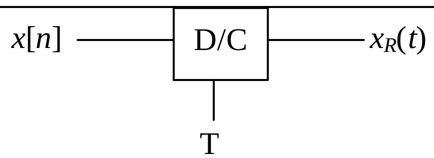
- $x_R(t) \Rightarrow$ Sinc-function interpolation of $x[n]$

$$= \frac{1}{2p}\int_{-\frac{p}{T_R}}^{\frac{p}{T_R}} T_R \hat{X}(\Omega T_R)\, e^{j\Omega t}\, d\Omega \;;\forall t \;\textbf{(R1)}\; \underset{m=\Omega T_R}{\underset{\uparrow}{=}} \frac{1}{2p}\int_{-p}^{p} \hat{X}(m)\, e^{j\frac{m}{T_R}t}\, dm$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{p}{T}(t-nT)}{\frac{p}{T}(t-nT)} \;;\forall t \quad \textbf{(R2)}$$

= $x_c(t)$ if $T <$ Nyquist interval for $x_c(t)$

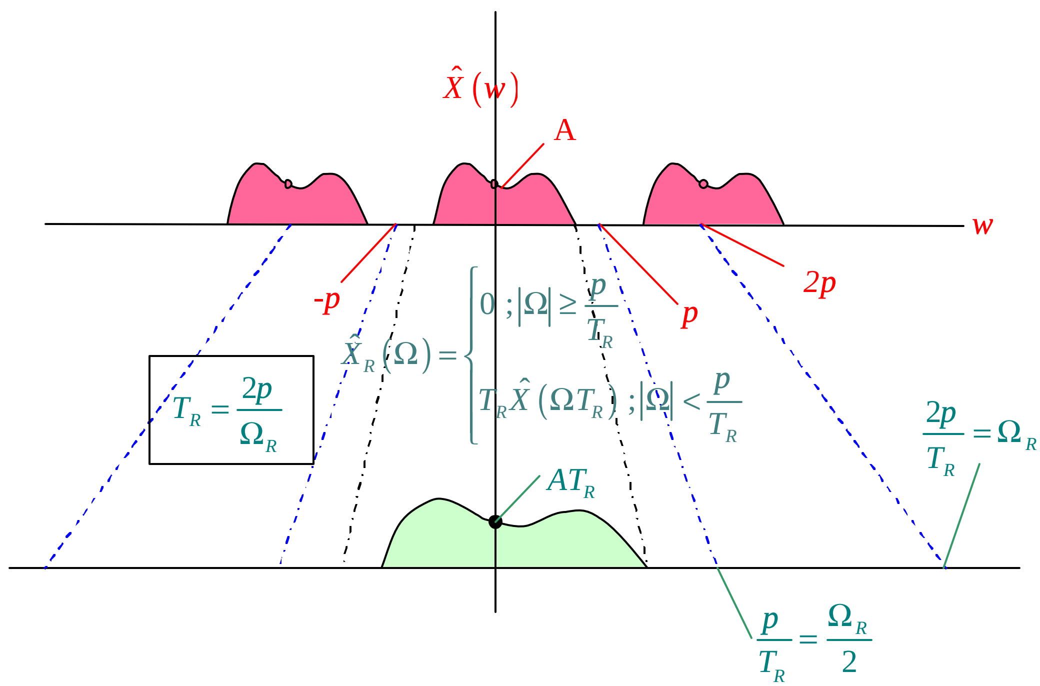
= the most parsimonious continuous time explanation for $x[n]$




- $x_R(nT) = x[n]$
- $$\hat{X}_R(\Omega) = T\hat{X}(w = \Omega T)p_{\frac{p}{T}}(\Omega) = \begin{cases} 0 & ;|\Omega| \geq \frac{p}{T} \\ T\hat{X}(\Omega T) & ;|\Omega| < \frac{p}{T} \end{cases}$$

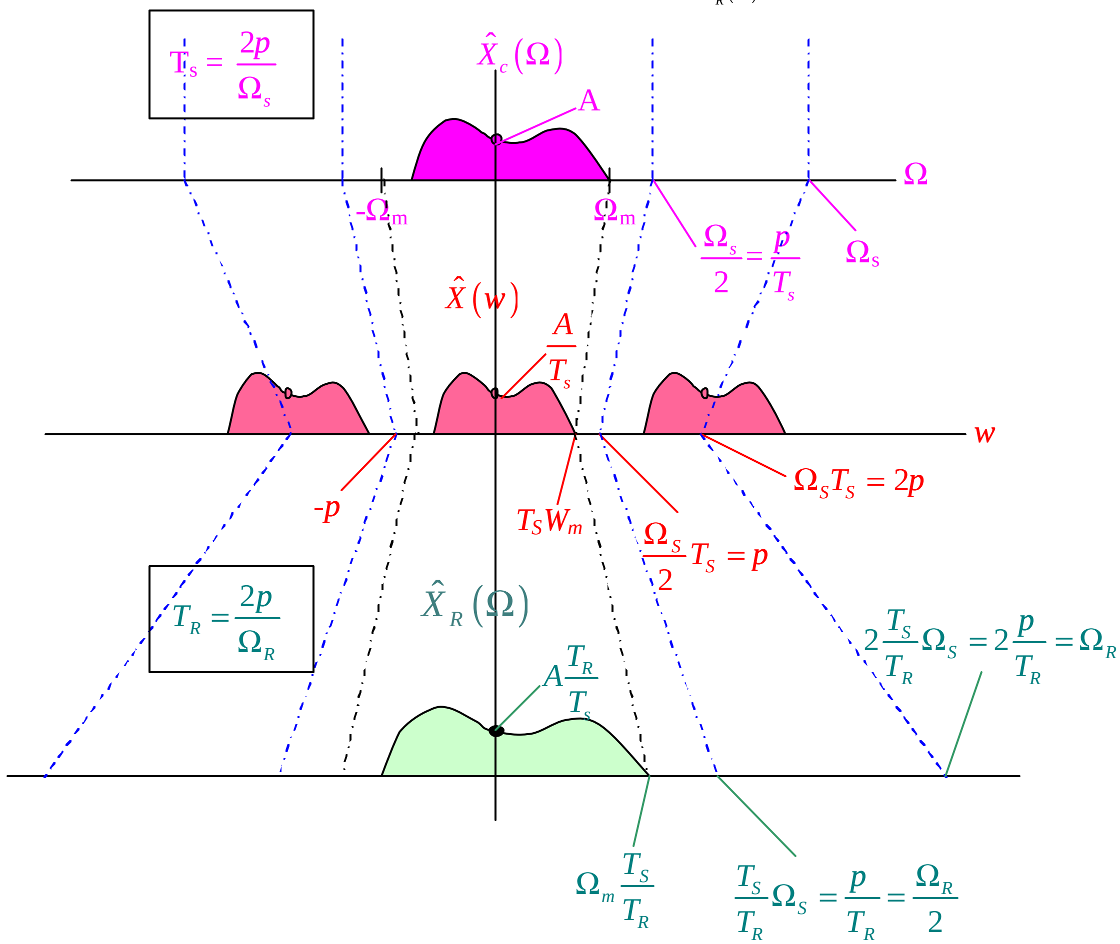
$x_R(t)$ is the unique continuous-time signal that has both properties

- $$\hat{X}_R(\Omega) = T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=\Omega T}$$



Proof $\quad \lim_{t\to nT} \dfrac{\sin\frac{\pi}{T}(t-mT)}{\frac{\pi}{T}(t-mT)} = \begin{cases} 0 & \text{for m} \neq \text{n} \\ 1 & \text{for m = n} \end{cases}$

Proof $\quad x_R(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} T\hat{X}(\Omega T)e^{j\Omega t}d\Omega = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left(T\hat{X}(\Omega T)p_{\frac{\pi}{T}}(\Omega)\right)}_{\hat{X}_R(\Omega)}e^{j\Omega t}d\Omega$




| | | | | |
|---|---|---|---|---|
| $\hat{X}_c(\Omega)$ | A | $\Omega_0$ | $\frac{\Omega_s}{2}$ | $\Omega_s$ |
| $\hat{X}(w)$ | $A\frac{1}{T_s}$ | $\omega_0 = \Omega_0 T_s$ | $\pi$ | $2\pi$ |
| $\hat{X}_R(\Omega)$ | $A\frac{T_R}{T_s}$ | $\frac{w_0}{T_R} = \Omega_0\frac{T_s}{T_R}$ | $\frac{\Omega_R}{2}$ | $\Omega_R$ |

- Note: The amplitude doesn't really change when sampling followed by reconstruction (under no-aliasing assumption determined by $T_s$). This doesn't depend on the choice of $T_R$ nor $T_s$. $x_R(t)$ will be equal to $x_c(t)$ if $T_s = T_R$. If $T$'s are different, $x_R(t)$ will be

$x_c(t)$ but expanded or shrinked in the time domain with all the height remains unchanged.

Can see this by the formula $x(at) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \frac{1}{|a|}\hat{X}\left(\frac{\Omega}{a}\right)$.

If no alising, $\hat{X}(w) = \frac{1}{T_S}\hat{X}_c\left(\frac{w}{T_S}\right) \Rightarrow \hat{X}_R(\Omega) = T_R\hat{X}(w = \Omega T_R) = \frac{T_R}{T_S}\hat{X}_c\left(\Omega\frac{T_R}{T_S}\right)$.

$$x\left(\frac{T_S}{T_R}t\right) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \frac{T_R}{T_S}\hat{X}_c\left(\Omega\frac{T_R}{T_S}\right)$$

## Ideal sampling

- **shah function** $\text{Ш}_T(t) = \sum_{n=-\infty}^{\infty} d(t - nT)$
- **$x_s(t)$** = $x_c(t)\sum_{n=-\infty}^{\infty} d(t - nT_S)$ = Ш-sampled version of $x_c(t)$ = $\sum_{n=-\infty}^{\infty} x[n]d(t - nT_S)$
- $\hat{X}_s(\Omega)$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT} = \hat{X}(w = \Omega T)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T_s}\hat{X}_c(\Omega - k\Omega_s);\ \Omega_s = \frac{2p}{T_s}$$

Proof $x_s(t) = \sum_{n=-\infty}^{\infty} x[n]d(t - nT_S)$

$$\hat{X}_s(\Omega) = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x[n]d(t - nT_S)e^{-j\Omega t}dt = \sum_{n=-\infty}^{\infty} x[n]\int_{-\infty}^{\infty} d(t - nT_S)e^{-j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT_S} = \hat{X}(w = \Omega T_S)$$

Know that $\hat{X}(w) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\hat{X}_c\left(\frac{w}{T} + k\frac{2p}{T}\right)\right)$.

Therefore, $\hat{X}_s(\Omega) = \hat{X}(w = \Omega T_S) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\hat{X}_c\left(\frac{\Omega T}{T} + k\frac{2p}{T}\right)\right)$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\left(\hat{X}_c(\Omega + k\Omega_S)\right)$$

- $x_S(t) \xrightarrow{\hat{H}_R(\Omega) = T\cdot p_{\Omega_m}(\Omega)} x_c(t)$ if $T < \frac{p}{\Omega_m}$
- For $W_0 > 2W_m \Rightarrow$ the shifted replicas don't overlap

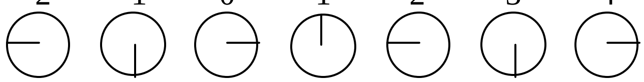
$$z(t) \xrightarrow{\hat{H}(\Omega)=T_0 \cdot p_{\Omega_m}(\Omega)} x(t)$$

## Wheel example: sampled scene sequence

- If sample at every T sec, move at constant speed,

  want to find $x_c(t) = e^{j\Omega_0 t}$.

  n= -2 -1 0 1 2 3 4



- Find the parsimonious $w_0$ first $\Rightarrow x[n]$ can = $e^{jw_0 n}$

  (Moving @ $w_0$ rad per 1 frame, in this case, $w_0 = \frac{p}{2}$)

- Can find $x_c(t) = e^{j\Omega_0 t}$ in two ways:
  - $x[n] = e^{jw_0 n} = e^{jw_0 n}\left(e^{j2p}\right)^{kn} = e^{j(w_0+2pk)n}$

    But $x[n] = x_c(nT) = e^{j\Omega_0 nT}$; thus $W_0 nT = (w_0+2pk)n \Rightarrow W_0 = \frac{w_0}{T} + 2p\frac{k}{T}$

  - Thinking in term of rev./sec:

    Fundamentally, moving @ $x = \frac{w_0}{2p}$ rev. per $T$ sec

    Can add k rev. more in 1 frame = $T$ sec.

    So, Possibly moving at $x+k$ rev. in $T$ sec

    $\frac{\Omega_0}{2p} = \frac{x}{T} + \frac{k}{T}$ rev/sec $\Rightarrow \Omega_0 = \frac{2px}{T} + 2p\frac{k}{T}$ rad/sec

- note that positive ω corresponds to a counterclockwise rotation
- Represented by $e^{j\left(2p\frac{x}{T}+2p\frac{k}{T}\right)t} = e^{j\left\langle 2p\frac{x}{T}\right\rangle t}$ where $-p \le \left\langle 2p\frac{x}{T}\right\rangle \le p$
- If $\left\langle 2p\frac{x}{T}\right\rangle \ne 2p\frac{x}{T}$
  - aliasing has occurred.
  - $e^{j\left(2p\frac{x}{T}+2p\frac{k}{T}\right)t}$ assumes the alias $e^{j\left\langle 2p\frac{x}{T}\right\rangle t}$

## One frequency example

- $x_c(t) = e^{j\Omega_0 t}$; $x[n] = e^{j\Omega_0 nT} = e^{jw_0 n} = e^{j(w_0+2pk)n}$; $W_0 = \frac{w_0}{T} + 2p\frac{k}{T}$
- $x_c(t) = e^{j\Omega_0 t} \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{X}(\Omega) = 2pd(\Omega - \Omega_0) \Rightarrow \Omega_0 = \Omega_m$

  $x[n] = e^{j\Omega_0 Tn} = e^{jw_0 n} = e^{j(w_0+2pk)n} \Rightarrow w_0 = \Omega_0 T - 2pk$

$$x[n] = e^{jnw_0} \overset{DTFT}{\rightleftharpoons} \hat{X}(w) = 2p \sum_{k=-\infty}^{\infty} d\left(w - \langle w_0 \rangle_{-p}^{p} + 2pk\right)$$

- $T_R = T_S = T$

$$x_R(t) = \frac{1}{2p} \int_{-p}^{p} 2pd\left(m - \langle w_0 \rangle_{-p}^{p}\right) e^{j\frac{m}{T}t}\, dm = e^{j\frac{\langle w_0 \rangle_{-p}^{p}}{T}t}$$

$$x_R(t) = \frac{1}{2p} \int_{-p}^{p} \hat{X}(m) e^{j\frac{m}{T}t}\, dm$$

If $\langle w_0 \rangle_{-p}^{p} = w_0$, $x_R(nT) = x[n]$

$$\hat{X}_R(\Omega) = 2pTd\left(\Omega T - \langle w_0 \rangle_{-p}^{p}\right) \xrightarrow{\Im^{-1}} x_R(t) = e^{j\frac{\langle w_0 \rangle_{-p}^{p}}{T}t} = e^{j\left(\langle \Omega_0 \rangle_{-\frac{\Omega_S}{2}}^{\frac{\Omega_S}{2}}\right)t}$$

$$\hat{X}_R(\Omega) = T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=\Omega T}$$

$$\frac{\langle w_0 \rangle_{-p}^{p}}{T} = \left\langle \frac{w_0}{T} \right\rangle_{-\frac{p}{T}}^{\frac{p}{T}} = \left\langle \frac{w_0}{T} \right\rangle_{-\frac{\Omega_S}{2}}^{\frac{\Omega_S}{2}} = \left\langle \frac{w_0}{T} \right\rangle_{-\frac{\Omega_S}{2}}^{\frac{\Omega_S}{2}}$$

From $w_0 = \Omega_0 T - 2pk$, $\left\langle \frac{w_0}{T} \right\rangle_{-\frac{\Omega_S}{2}}^{\frac{\Omega_S}{2}} = \left\langle \frac{\Omega_0 T - 2pk}{T} \right\rangle_{-\frac{\Omega_S}{2}}^{\frac{\Omega_S}{2}} = \left\langle \Omega_0 - k\frac{2p}{T} \right\rangle_{-\frac{\Omega_S}{2}}^{\frac{\Omega_S}{2}}$

$$= \langle \Omega_0 - k\Omega_S \rangle_{-\frac{\Omega_S}{2}}^{\frac{\Omega_S}{2}} = \langle \Omega_0 \rangle_{-\frac{\Omega_S}{2}}^{\frac{\Omega_S}{2}}$$

- $T_R \neq T_S$

$$x_R(t) = \frac{1}{2p} \int_{-p}^{p} 2pd\left(m - \langle w_0 \rangle_{-p}^{p}\right) e^{j\frac{m}{T_R}t}\, dm = e^{j\frac{\langle w_0 \rangle_{-p}^{p}}{T_R}t}$$

$$\hat{X}_R(\Omega) = 2pT_R d\left(\Omega T_R - \langle w_0 \rangle_{-p}^{p}\right) \xrightarrow{\Im^{-1}} x_R(t) = e^{j\frac{\langle w_0 \rangle_{-p}^{p}}{T_R}t} = e^{j\left(\left\langle \Omega_0 \frac{T_S}{T_R} \right\rangle_{-\frac{\Omega_R}{2}}^{\frac{\Omega_R}{2}}\right)t}$$

$$\frac{\langle w_0 \rangle_{-p}^{p}}{T_R} = \left\langle \frac{w_0}{T_R} \right\rangle_{-\frac{p}{T_R}}^{\frac{p}{T_R}} = \left\langle \frac{\Omega_0 T_S - 2pk}{T_R} \right\rangle_{-\frac{p}{T_R}}^{\frac{p}{T_R}} = \left\langle \Omega_0 \frac{T_S}{T_R} - \frac{2pk}{T_R} \right\rangle_{-\frac{p}{T_R}}^{\frac{p}{T_R}}$$

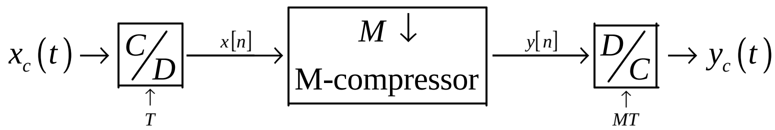
$$= \left\langle \Omega_0 \frac{T_S}{T_R} - k\Omega_R \right\rangle_{-\frac{\Omega_R}{2}}^{\frac{\Omega_R}{2}} = \left\langle \Omega_0 \frac{T_S}{T_R} \right\rangle_{-\frac{\Omega_R}{2}}^{\frac{\Omega_R}{2}}$$

## Downsampling

- $y[n] = x[nM]$ = ***M*-down sampled** version of $x[n]$

  = a “compressed” version of $x[n]$

  $$x[n] \xrightarrow[\text{M-compressor}]{M\downarrow} y[n]$$




  - $T' = MT$

- $$\hat{Y}(w) = \frac{1}{M}\sum_{\ell=0}^{M-1} \hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right)$$

- Let $$z[n] = \begin{cases} x[n]; \text{ if } \frac{n}{M} \in I \\ 0; \text{ if } \frac{n}{M} \notin I \end{cases}$$

Then, can rewrite $z[n]$ as $z[n] = \left(\frac{1}{M}\sum_{\ell=0}^{M-1} e^{j2p\ell\frac{n}{M}}\right)x[n] = \frac{1}{M}\sum_{\ell=0}^{M-1} x[n]e^{j2p\ell\frac{n}{M}}$ .

To see this, note that $$\frac{1}{M}\sum_{\ell=0}^{M-1} e^{j2p\ell\frac{n}{M}} = \begin{cases} 1; \text{ if } \frac{n}{M} \in I \\ 0; \text{ if } \frac{n}{M} \notin I \end{cases}$$

$$\sum_{\ell=0}^{M-1} e^{j2p\ell\frac{n}{M}} = \sum_{\ell=0}^{M-1}\left(e^{j2p\frac{n}{M}}\right)^{\ell} = \frac{1 - e^{j2p\frac{n}{M}M}}{1 - e^{j2p\frac{n}{M}}} = \frac{1 - e^{j2pn}}{1 - e^{j2p\frac{n}{M}}}$$

$$= 0 \text{ if } \frac{n}{M} \notin I \text{ since } e^{j2p\frac{n}{M}} \neq 1$$

If $\frac{n}{M} \in I$, $$\sum_{\ell=0}^{M-1} e^{j2p\ell\frac{n}{M}} = \frac{1 - \left(e^{j2p}\right)^{n}}{1 - \left(e^{j2p}\right)^{\frac{n}{M}}}$$

$$= \frac{1 - \left(e^{j2p}\right)^{n}}{1 - \left(e^{j2p}\right)^{\frac{n}{M}}} = \lim_{x\to1}\frac{1 - x^{n}}{1 - x^{\frac{n}{M}}} = \lim_{x\to1}\frac{-nx^{n-1}}{-\frac{n}{M}x^{\frac{n}{M}-1}} = M$$

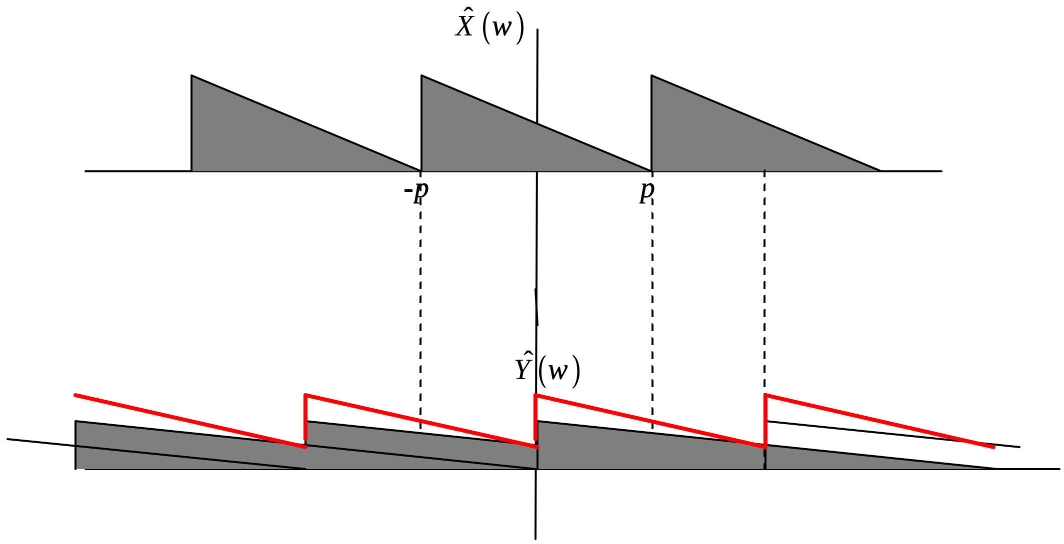
From $z[n] = \frac{1}{M}\sum_{\ell=0}^{M-1} x[n]e^{j2p\ell\frac{n}{M}}$, we have $\boxed{\hat{Z}(w) = \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(w - \ell\frac{2p}{M}\right)}$ from frequency-shift rule. $\Rightarrow$ aliasing is possible

- Let $y[n] = z[nM] = x[nM]$

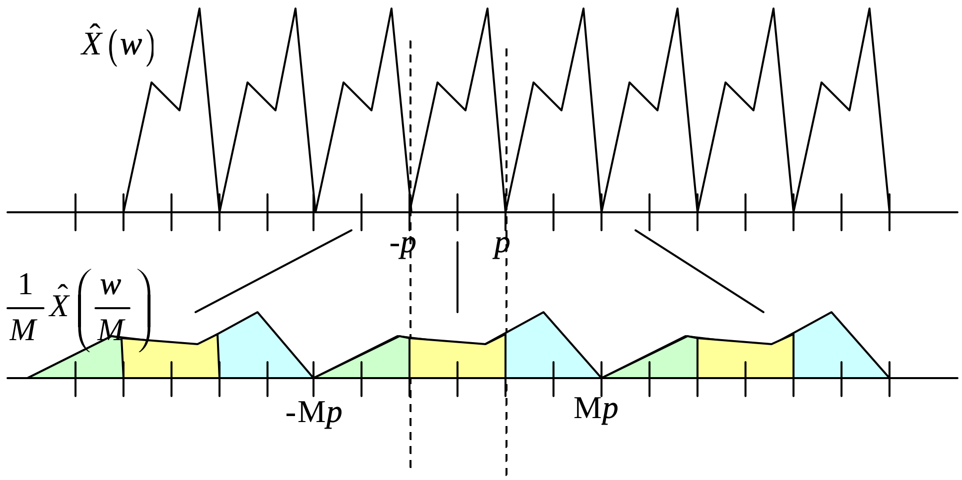
  Then $\hat{Z}(w) = \sum_{m=-\infty}^{\infty} z[m]e^{-jmw} = \sum_{n=-\infty}^{\infty} z[nM]e^{-jnMw} = \sum_{n=-\infty}^{\infty} y[n]e^{-jnMw} = \hat{Y}(Mw)$

  - $\hat{Y}(w)$ is an M-expanded version of $\hat{Z}(w)$

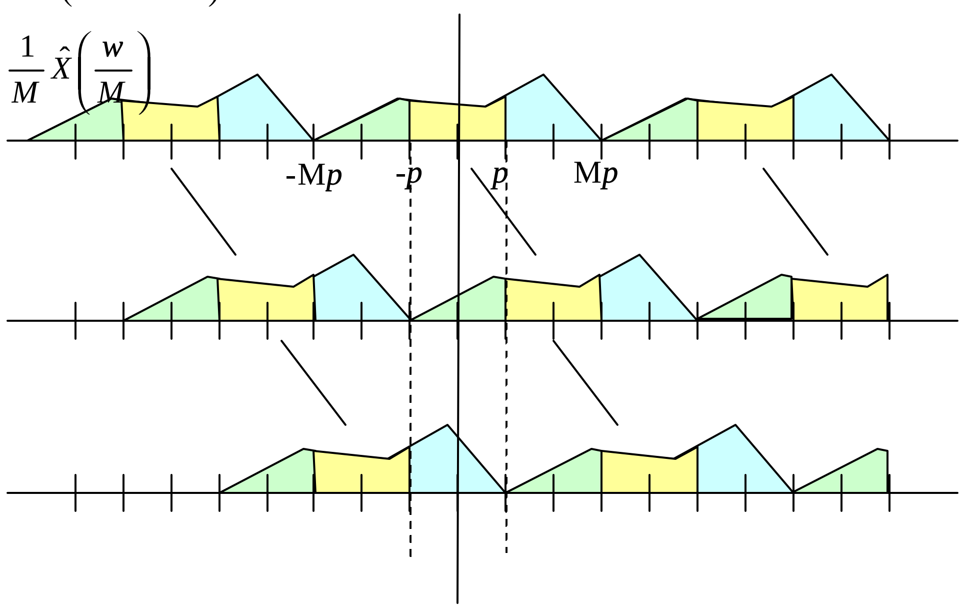
  Or $\boxed{\hat{Y}(w) = \hat{Z}\left(\frac{w}{M}\right)} = \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right)$




- Now, let's take a closer look at $\hat{Y}(w) = \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right)$. It is a summation of $\frac{1}{M}\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right)$. Each $\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right)$ is an expanded and shifted version of $\hat{X}(w)$. Since $\hat{Y}(w)$ is 2π-periodic, we will try to find what part of $\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right)$ falls in the -π to π range. First, note that the region from -π to π of $\hat{X}(w)$ will be expanded to the range –Mπ to Mπ as shown in the figure below:

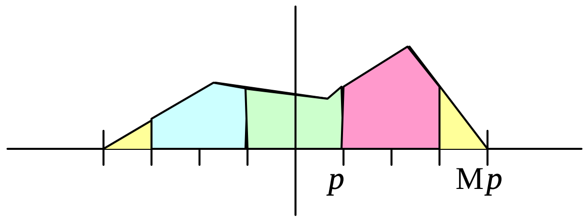



Each of the $\frac{1}{M}\hat{X}\left(\frac{w}{M}-\ell\frac{2p}{M}\right)$ is indeed $\frac{1}{M}\hat{X}\left(\frac{w}{M}\right)$ shifted by $2p\ell$ $\left(\frac{w}{M}-\ell\frac{2p}{M}=0\Rightarrow w=2p\ell\right)$. Thus, $\hat{Y}(w)$ is the summation of all the $\frac{1}{M}\hat{X}\left(\frac{w}{M}-\ell\frac{2p}{M}\right)$ as shown below:




Looking at only the part of $\hat{X}\left(\frac{w}{M}-\ell\frac{2p}{M}\right)$ which falls in the $-\pi$ to $\pi$ range, we see that $\hat{X}\left(\frac{w}{M}\right)$ is partitioned into M pieces, each pieces width equal $2\pi$. $\hat{Y}(w)$ is the summation of all these pieces times $\frac{1}{M}$. Therefore, $\hat{Y}(w)$ is basically an average of all M pieces of $\hat{X}\left(\frac{w}{M}\right)$.

Note that if M is even, then the first land last π chunks of $\hat{X}\left(\frac{w}{M}\right)$ construct one 2π piece.




- $\hat{Y}(w+2kp)=\hat{Y}(w)$

  To see this, note that we want to have

  $$\frac{1}{M}\sum_{\ell'=0}^{M-1}\hat{X}\left(\frac{w+2kp}{M}-\ell'\frac{2p}{M}\right)=\frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w}{M}-\ell\frac{2p}{M}\right)$$

  and that $\hat{X}\left(\frac{w+2kp}{M}-\ell'\frac{2p}{M}+2p\,n\right)=\hat{X}\left(\frac{w+2kp}{M}-\ell'\frac{2p}{M}\right)$.

  We will show that, given $k$, there exist one and only one integer $\ell'$ for each $\ell$ that could make $\hat{X}\left(\frac{w+2kp}{M}-\ell'\frac{2p}{M}+2p\,n\right)=\hat{X}\left(\frac{w}{M}-\ell\frac{2p}{M}\right)$, using appropriate $n$.

  To have $\frac{w+2kp}{M}-\ell'\frac{2p}{M}+2p\,n=\frac{w}{M}-\ell\frac{2p}{M}$, need

  $k-\ell'+nM=-\ell$ or $n=\frac{\ell'-\ell-k}{M}$.

  Thus, given $k$, to find which $\ell'\in\{0,1,\ldots,M-1\}$ or which term of $\hat{X}\left(\frac{w+2kp}{M}-\ell'\frac{2p}{M}+2p\,n\right)$ will be equal to $\hat{X}\left(\frac{w}{M}-\ell_0\frac{2p}{M}\right)$, we need to find $\ell'$ which give $n=\frac{\ell'-\ell_0-k}{M}$ an integer value. There is one and only one $\ell'$ that could do this, because $0\le\ell'\le M-1$. Only one of $\ell'$ will give $(-\ell_0-k)+\ell'$ that is divisible by M. This yields cyclic mapping between $\ell$ and $\ell'$, and thus each term of $\hat{X}\left(\frac{w+2kp}{M}-\ell'\frac{2p}{M}\right)$'s is equal to one of $\hat{X}\left(\frac{w}{M}-\ell\frac{2p}{M}\right)$. And therefore, the sum is equal.
- Example:

$$\hat{Y}(w-2p) = \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w-2p}{M} - \ell\frac{2p}{M}\right) = \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w}{M} - \frac{2p}{M} - \ell\frac{2p}{M}\right)$$

$$= \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w}{M} - (\ell+1)\frac{2p}{M}\right) = \frac{1}{M}\sum_{k=1}^{M}\hat{X}\left(\frac{w}{M} - k\frac{2p}{M}\right)$$

$$= \frac{1}{M}\left(\sum_{k=1}^{M-1}\hat{X}\left(\frac{w}{M} - k\frac{2p}{M}\right) + \hat{X}\left(\frac{w}{M} - 2p\right)\right)$$

$$= \frac{1}{M}\left(\sum_{k=1}^{M-1}\hat{X}\left(\frac{w}{M} - k\frac{2p}{M}\right) + \hat{X}\left(\frac{w}{M}\right)\right) = \frac{1}{M}\sum_{k=0}^{M-1}\hat{X}\left(\frac{w}{M} - k\frac{2p}{M}\right) = \hat{Y}(w)$$

- $\hat{Y}(w) = \frac{1}{MT}\sum_{k=-\infty}^{\infty}\left(\hat{X}_c\left(\frac{w}{MT} + k\frac{2p}{MT}\right)\right)$

Think about going directly from $x_c(t)$ to y[$n$], then

- $\hat{Y}(w) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T'}\hat{X}_c\left(\frac{w}{T'} + k\frac{2p}{T'}\right)\right)$. Here $T' = MT$. Therefore,

  $$\hat{Y}(w) = \frac{1}{MT}\sum_{k=-\infty}^{\infty}\left(\hat{X}_c\left(\frac{w}{MT} + k\frac{2p}{MT}\right)\right)$$

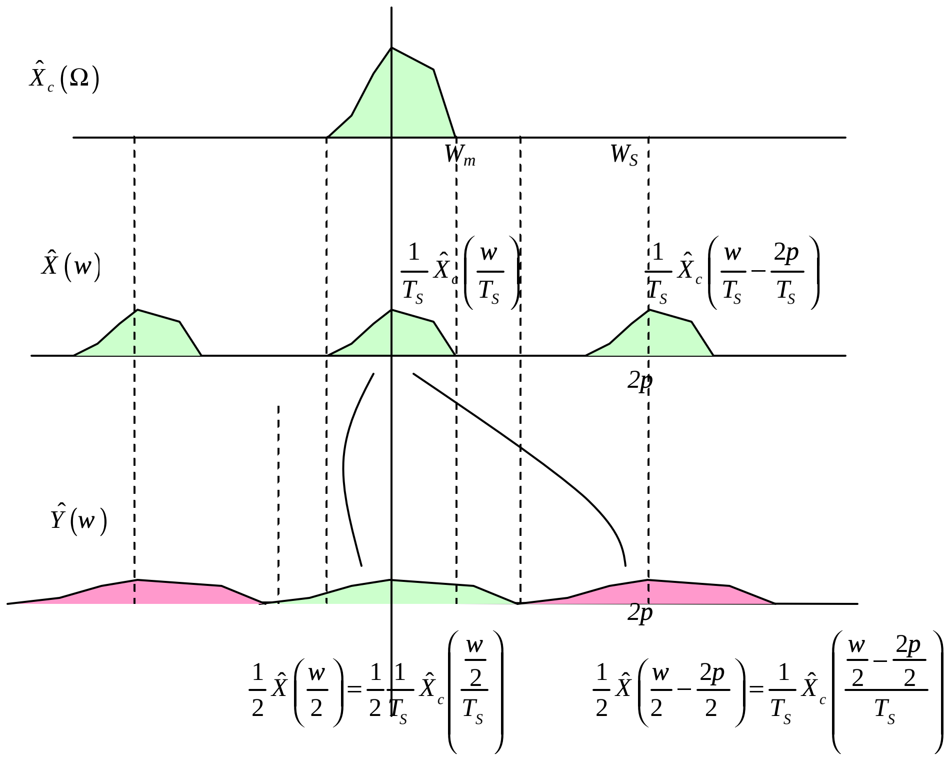
- Compare this to $\hat{Y}(w) = \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right)$. We know that $\hat{X}(w) = \frac{1}{T}\hat{X}_c\left(\frac{w}{T}\right)$ if no aliasing.

  Thus, $\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right) = \frac{1}{T}\hat{X}_c\left(\frac{\frac{w}{M} - \ell\frac{2p}{M}}{T}\right) = \frac{1}{T}\hat{X}_c\left(\frac{w}{MT} - \ell\frac{2p}{MT}\right)$, and

  $\hat{Y}(w) = \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right) = \frac{1}{MT}\sum_{\ell=0}^{M-1}\hat{X}_c\left(\frac{w}{MT} - \ell\frac{2p}{MT}\right)$ same result.

- If aliasing, still, $\hat{Y}(w) = \frac{1}{MT}\sum_{k=-\infty}^{\infty}\left(\hat{X}_c\left(\frac{w}{MT} + k\frac{2p}{MT}\right)\right)$.

  This is easy to see since $y[n] = x[Mn] = x(nMT)$. Can get $y[n]$ but just sampling $x(t)$ @ $MT$ period.




- Example (doing it directly)
  - $y[n] = x[2n]$

$$\hat{Y}(w)=\sum_{n=-\infty}^{\infty} y[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} x[2n]e^{-jwn}$$

Be careful here and notice that $\sum_{n=-\infty}^{\infty} x[2n]e^{-jwn} \neq \sum_{m=-\infty}^{\infty} x[m]e^{-jw\frac{m}{2}}$.

$$\sum_{n=-\infty}^{\infty} x[2n]e^{-jwn} = \ldots + x[0] + x[2]e^{-jw1} + x[4]e^{-jw2} + \ldots$$

$$\sum_{m=-\infty}^{\infty} x[m]e^{-jw\frac{m}{2}} = \ldots + \boxed{x[0]} + x[1]e^{-jw\frac{1}{2}} + \boxed{x[2]e^{-jw1}} + \ldots$$

We want $\sum_{m=-\infty}^{\infty} x[m]e^{-jw\frac{m}{2}}$ but only want the even term.

Use $\frac{1}{M}\sum_{\ell=0}^{M-1} e^{j2p\ell\frac{n}{M}} = \begin{cases} 1;\ \text{if } \frac{n}{M}\in I \\ 0;\ \text{if } \frac{n}{M}\notin I \end{cases}$

$$\Rightarrow \frac{1}{2}\sum_{\ell=0}^{1} e^{j2p\ell\frac{m}{2}} = \begin{cases} 1;\ \text{if } \frac{m}{2}\in I \\ 0;\ \text{if } \frac{m}{2}\notin I \end{cases} = \begin{cases} 1;\ \text{if } m \text{ even} \\ 0;\ \text{if } m \text{ odd} \end{cases}$$

$$\frac{1}{2}\sum_{\ell=0}^{1} e^{j2p\ell\frac{m}{2}} = \frac{1}{2}\sum_{\ell=0}^{1} e^{jp\ell m} = \frac{1}{2}\left(e^{0} + e^{jpm}\right) = \frac{1}{2}\left(1+(-1)^{m}\right) = \begin{cases} 1; \text{ if } m \text{ even} \\ 0; \text{ if } m \text{ odd} \end{cases}$$

Thus, $\hat{Y}(w) = \sum_{n=-\infty}^{\infty} x[2n]e^{-jwn} = \sum_{m=-\infty}^{\infty} \frac{1}{2}\left(e^{0} + e^{jpm}\right)x[m]e^{-jw\frac{m}{2}}$

$$= \frac{1}{2}\sum_{m=-\infty}^{\infty} x[m]e^{-jm\frac{w}{2}} + \frac{1}{2}\sum_{m=-\infty}^{\infty} x[m]e^{-j\left(\frac{w}{2}-p\right)m}$$

$$= \frac{1}{2}\hat{X}\left(\frac{w}{2}\right) + \frac{1}{2}\hat{X}\left(\frac{w}{2}-p\right)$$

Same as using the formula $\hat{Y}(w) = \frac{1}{M}\sum_{\ell=0}^{M-1}\hat{X}\left(\frac{w}{M} - \ell\frac{2p}{M}\right)$, letting $M = 2$

- Basically, this is $x_c(t)$, sampled @ $T' = MT \Rightarrow$ To recover $x_c(t)$ from $y_c(t)$ completely, need $MT < \frac{p}{\Omega_m}$ $\Rightarrow$more stringent

## Upsampling

- 2-step process:

  $x[n] \longrightarrow \boxed{D/C} \longrightarrow q(t) \longrightarrow \boxed{C/D} \longrightarrow y[n]$, with $T$ for D/C and $T' = \frac{T}{L}$ for C/D

  - So, $y[n]$ is $x[n]$ added with the parsimonious approximation, using the information from $x[n]$

- $y[n] = \sum_{m=-\infty}^{\infty} x[m] \frac{\sin p\left(\frac{n}{L} - m\right)}{p\left(\frac{n}{L} - m\right)}$

- $\hat{Y}(w) = L\hat{X}(Lw)\, p_{\frac{p}{L}}(w) = \begin{cases} L\hat{X}(wL) & ; |w| < \frac{p}{L} \\ 0 & ; |w| \geq \frac{p}{L} \end{cases}$ for $|w| \leq p$

  - Replicate $L\hat{X}(Lw)$ over each $w = k2\pi$

- $T$ doesn't matter
- View 1

  $x[n] \longrightarrow \boxed{D/C} \longrightarrow q(t) \longrightarrow \boxed{C/D} \longrightarrow y[n]$, with $T$ for D/C and $T' = \frac{T}{L}$ for C/D

Note that we have two sets of variables:

$x_c(t), \hat{X}(\Omega), x[n], \hat{X}(w), x_R(t), \hat{X}_R(\Omega)$

$q(t), \hat{Q}(\Omega), y[n], \hat{Y}(w)$

- $$\boxed{q(t) = x_R(t) = \sum_{m=-\infty}^{\infty} x[m] \frac{\sin \frac{p}{T}(t - mT)}{\frac{p}{T}(t - mT)}}$$

- $$y[n] = q(nT') = q\left(n\frac{T}{L}\right) = \sum_{m=-\infty}^{\infty} x[m] \frac{\sin p\left(\frac{n}{L} - m\right)}{p\left(\frac{n}{L} - m\right)}$$

- From $\hat{X}_R(\Omega) = T\hat{X}(w = \Omega T)\, p_{\frac{p}{T}}(\Omega)$,

  $$(\Omega_m)_{\hat{Q}} = (\Omega_m)_{\hat{X}_R} = \frac{p}{T}$$

- $y[n]$ is a sampled version of $q(t) = x_R(t)$. Thus,

  $$\hat{Y}(w) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T'}\hat{Q}\left(\frac{w}{T'} + k\frac{2p}{T'}\right)\right) = \sum_{k=-\infty}^{\infty}\left(\frac{L}{T}\hat{Q}\left(\frac{Lw}{T} + k\frac{2pL}{T}\right)\right)$$

  No overlap between $\frac{L}{T}\hat{Q}\left(\frac{Lw}{T} + k\frac{2pL}{T}\right)$ if $\Omega_S' \geq 2(\Omega_m)_{\hat{Q}} \Rightarrow \frac{2p}{T'} \geq 2\frac{p}{T} \Rightarrow T' \leq T$

  If $L > 1$, $T' = \frac{T}{L} \leq T$ , definitely no overlap,

  and $\boxed{\hat{Y}(w) = \frac{L}{T}\hat{Q}\left(\frac{Lw}{T}\right) = \frac{L}{T}\hat{X}_R\left(\frac{Lw}{T}\right) \text{ for } -p \leq w \leq p}$

  - replicate $\frac{L}{T}\hat{X}_R\left(\frac{wL}{T}\right)$ over each $w = k2\pi$

  Since $\hat{X}_R(\Omega) = T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=\Omega T}$ ,

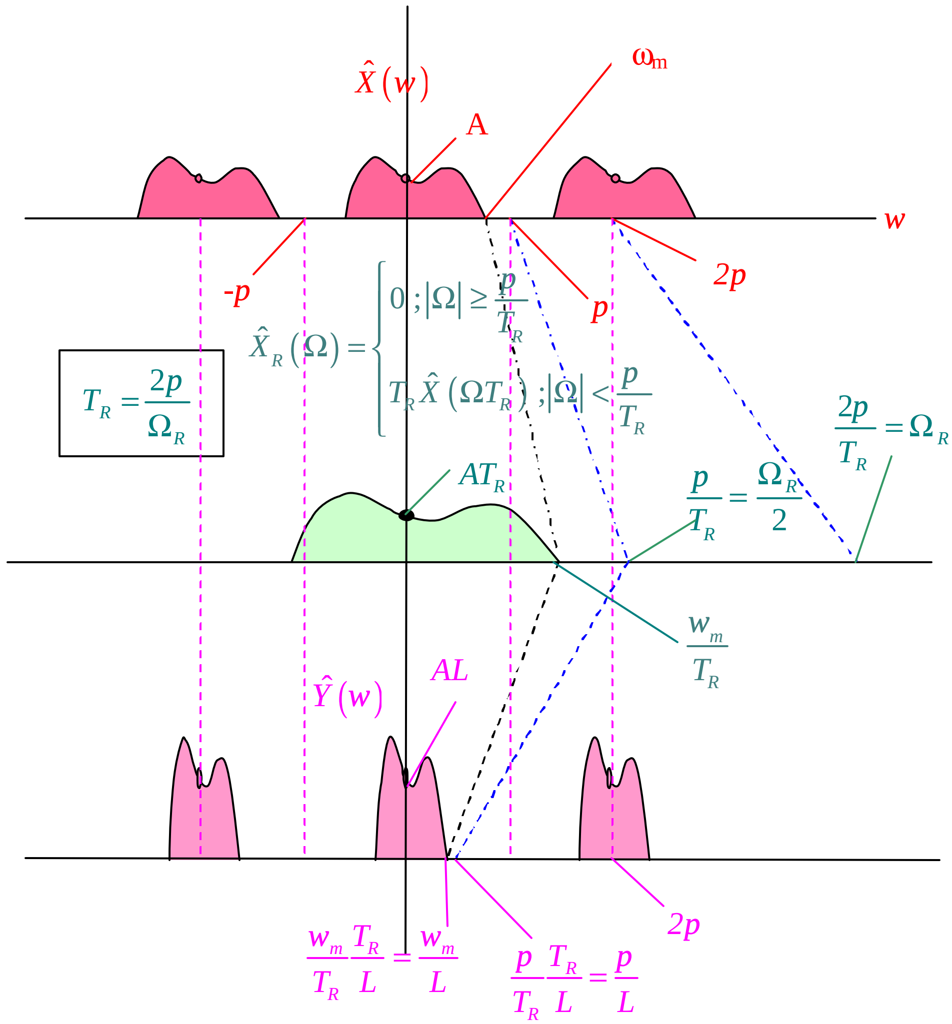
  $$\hat{X}_R\left(\frac{Lw}{T}\right) = T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=\frac{Lw}{T}T} = T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=Lw}$$

  $$= T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=Lw}$$

  $$\hat{Y}(w) = \frac{L}{T}\hat{X}_R\left(\frac{Lw}{T}\right) = \frac{L}{T}T\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=Lw} = L\left(\hat{X}(w)\, p_p(w)\right)\Big|_{w=Lw}$$

  for $-p \leq w \leq p$

Thus, for $|w| \le p$  $\hat{Y}(w) = L\hat{X}(Lw)\, p_{\frac{p}{L}}(w) = \begin{cases} L\hat{X}(wL) & ; |w| < \frac{p}{L} \\ 0 & ; |w| \ge \frac{p}{L} \end{cases}$




- View 2

  - $z_u[n]$ = $L$-expanded $x[n]$ = $\begin{cases} x\left[\frac{n}{L}\right] & ; \frac{n}{L} \in I \\ 0 & ; \frac{n}{L} \notin I \end{cases}$

  - $\hat{Z}(w) = \hat{X}(Lw) = \sum_{k=-\infty}^{\infty} \left( \frac{1}{T} \hat{X}_R\left( \frac{wL}{T} + k\frac{2p}{T} \right) \right)$

    $\Rightarrow$ replicate $\frac{1}{T}\hat{X}_R\left(\frac{wL}{T}\right)$ over each $\omega = k\frac{2p}{L}$

$$\hat{Z}(w) = \sum_{n=-\infty}^{\infty} z[n] e^{-jwn} = \sum_{\substack{n=-\infty \\ \frac{n}{L} \in I}}^{\infty} z[n] e^{-jwn} = \sum_{m=-\infty}^{\infty} z[Lm] e^{-jwLm}$$

$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j(Lw)m}$$

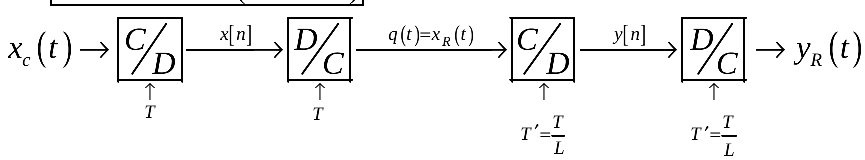
$$\boxed{\hat{Z}(w) = \hat{X}(Lw)}$$

- Example $z[n] = \begin{cases} x\left[\frac{n}{2}\right] & \text{n even} \\ 0 & \text{n odd} \end{cases}$

$$\hat{Z}(w) = \sum_{n=-\infty}^{\infty} z[n] e^{-jwn} = \sum_{\substack{n=-\infty \\ \text{n even}}}^{\infty} z[n] e^{-jwn} = \sum_{m=-\infty}^{\infty} z[2m] e^{-jw2m}$$

$$= \sum_{m=-\infty}^{\infty} x[m] e^{-j(2w)m} = \hat{X}(2w)$$

- $\boxed{\hat{Y}(w) = \hat{Z}(w) \cdot \left( Lp_{\frac{p}{L}}(w) \right)}$ for $|w| \le p$

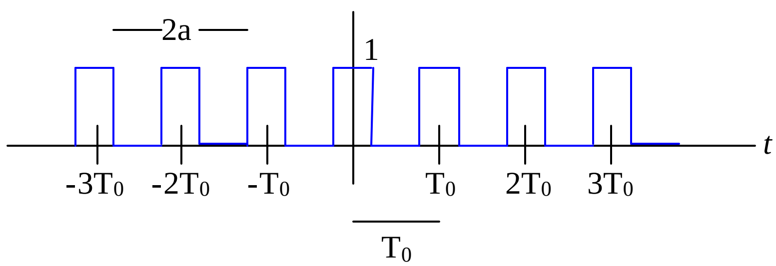
- 


- If $T \le \frac{p}{\Omega_m}$, $q(t) = x_c(t)$ and $y_R(t) = x_c(t)$ because $\frac{T}{L} < T$

## Practical sampling

- $\boxed{r(t) = \sum_{n=-\infty}^{\infty} p_a(t - nT_0) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} \left( \frac{\sin(akw_0)}{kp} \right) e^{jkw_0 t}\ ;\ c_0 = \frac{2a}{T_0};\ \Omega_0 = \frac{2p}{T_0}}$

  - Rectangular pulse train; height = 1
    Each pulse width = 2a
    Centered at 0, $\pm T_0$, $\pm 2T_0$, …




$T_0 >$ small $a > 0$

- $= \sum_{n=-\infty}^{\infty} p_a(t - nT_0)$
- $= \sum_{k=-\infty}^{\infty} \left( \frac{\sin(ak\Omega_0)}{kp} \right) e^{jk\Omega_0 t}$

Proof $\quad c_k = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} r(t) e^{-jk\Omega_0 t} dt = \int_{-a}^{a} e^{-jk\Omega_0 t} dt = \frac{\sin(ak\Omega_0)}{kp}$

Proof $\quad c_0 = \lim_{k \to 0} \frac{\sin(ak\Omega_0)}{kp} = \frac{a\Omega_0}{p} \lim_{k \to 0} \frac{\sin(ak\Omega_0)}{a\Omega_0 k} = \frac{a\Omega_0}{p} 1 = \frac{a2\not{p}}{\not{p} T_0} = \frac{2a}{T_0}$

Proof 2 $\quad$ Can see from the graph that $E[r(t)] = \frac{2a}{T} = c_0$.

- $\hat{R}(\Omega) = \sum_{k=-\infty}^{\infty} 2p\, c_k d(\Omega - k\Omega_0) = \sum_{k=-\infty}^{\infty} \frac{2\sin(ak\Omega_0)}{k} d(\Omega - k\Omega_0)$

To see this, recall that $r(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t} \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{R}(\Omega) = \sum_{k=-\infty}^{\infty} 2p\, c_k d(\Omega - k\Omega_0)$

- $z(t)$ = a **practically sampled** version of $x(t) = x(t)r(t)$

each width = 2a

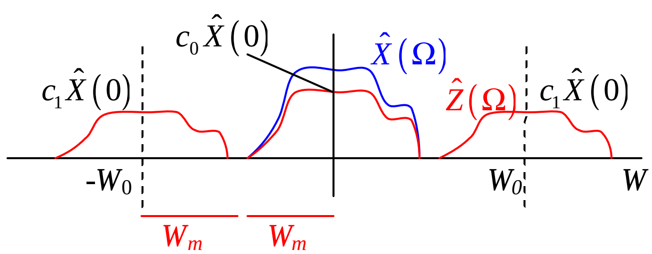
centered at 0, $\pm T_0$, $\pm 2T_0$, …

height = 0 or height of $x(t)$

- $\hat{Z}(\Omega) = \sum_{k=-\infty}^{\infty} c_k \hat{X}(\Omega - k\Omega_0)$; $\Omega_0 = \frac{2p}{T_0}$

$\hat{Z}(\Omega)$ = the sum of scaled shifted replicas of $\hat{X}(\Omega)$

$c_k$ = the $k^{th}$ fourier coefficient of the pulse train $r(t)$




Proof $\quad z(t) = x(t) r(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{Z}(\Omega) = \frac{1}{2p} \hat{X}(\Omega) * \hat{R}(\Omega)$

$$\hat{Z}(\Omega)=\frac{1}{2p}\hat{X}(\Omega)*\hat{R}(\Omega)=\frac{1}{\cancel{2p}}\hat{X}(\Omega)*\left(\sum_{k=-\infty}^{\infty}\cancel{2p}\,c_k d(\Omega-k\Omega_0)\right)$$

$$=\sum_{k=-\infty}^{\infty}c_k\hat{X}(\Omega)*d(\Omega-k\Omega_0)=\sum_{k=-\infty}^{\infty}c_k\hat{X}(\Omega-k\Omega_0)$$

- For $W_0 > 2W_m \Rightarrow$

  the shifted replicas don't overlap

  $$z(t)\xrightarrow{\hat{H}(\Omega)=p_{\Omega_m}(\Omega)}c_0 x(t)=\frac{2a}{T_0}x(t)$$

- If define $r(t)=\sum_{n=-\infty}^{\infty}\frac{1}{a}p_{\frac{a}{2}}(t-nT)$
  - $\lim_{a\to 0} r(t)=\text{Ш}_T(t)$
  - $x_s(t)=\lim_{a\to 0} r(t)x_c(t)$
  - $x_s(t)=\sum_{n=-\infty}^{\infty}x[n]d(t-nT)$

## Practical reconstruction; Interpolation

- $x_R(t)=\sum_{n=-\infty}^{\infty}x[n]h_R(t-nT)=h_R(t)*\left(\sum_{n=-\infty}^{\infty}x[n]d(t-nT)\right)=h_R(t)*x_s(t)$

- $\hat{X}_R(\Omega)=\hat{H}_R(\Omega)\cdot\hat{X}_s(\Omega)=\hat{H}_R(\Omega)\cdot\sum_{k=-\infty}^{\infty}\frac{1}{T_s}\hat{X}_c(\Omega-k\Omega_s)$

- $h_R(t)$ = **interpolating function**

- Staircase or zero-order hold interpolation

  $$h_{\square}(t)=p_{\frac{T}{2}}(t)\underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}}\hat{H}(\Omega)=\frac{2\sin\left(\Omega\frac{T}{2}\right)}{\Omega}$$

  $\Rightarrow x_R(t)=x(nT)$ for $nT-\frac{T}{2}<t<nT+\frac{T}{2}$

  - $\underbrace{x[n]}_{\substack{\uparrow\\ \text{constant}}}d(t-nT)*p_{\frac{T}{2}}(t)=x[n]\,p_{\frac{T}{2}}(t-nT)$

  - $\hat{H}_R(\Omega)=0$ when $\Omega=k\Omega_s\Rightarrow$ killed high-freq replicas

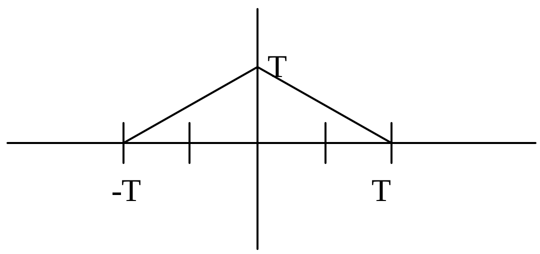
    $$\left(\hat{X}_s(\Omega)=\sum_{k=-\infty}^{\infty}\frac{1}{T_s}\hat{X}_c(\Omega+k\Omega_S)\right)$$

- $\hat{H}_R(\Omega) = T$ when $\Omega \to 0 \Rightarrow$ cancel $\frac{1}{T}$ from $\hat{X}_s(\Omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} \hat{X}_c(\Omega + k\Omega_S)$

- Linear interpolation : connect-the-dot

$$h_\Delta(t) = \frac{1}{T} h_\square(t) * h_\square(t) \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{H}_\Delta(\Omega) = \frac{1}{T}\left(\frac{2\sin\left(\Omega\frac{T}{2}\right)}{\Omega}\right)^2$$

- $p_{\frac{T}{2}}(t) * p_{\frac{T}{2}}(t) = \begin{cases} 0 & |t| \ge T \\ T - |t| & 0 \le |t| < T \end{cases}$




- $h(t) = T_0 \frac{\sin\left(\frac{w_0}{2} t\right)}{p t} \underset{\Im^{-1}}{\overset{\Im}{\rightleftharpoons}} \hat{H}(w) = T_0 p_{\frac{w_0}{2}}(w)$