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INTONATION VARIABLES IN THE PERFORMANCE OF TWELVE-TONE MUSIC

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"Intonation Variables in the Performance of Twelve-Tone Music"

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The possibility of intonation in twelve-tone music being preferentially variable rather than adhering strictly to an equal duodecimal division of the octave corresponding to equal temperament is considered in the light of findings that the intonation of string players and others tends to exhibit characteristics of Pythagorean tuning rather than of equal temperament. The nature of Pythagorean intonation with its enharmonic pitch distinctions (involving the Pythagorean comma) is discussed along with pitch adjustments (involving the syntonic comma, diaskhisma, diesis, etc.) sometimes required for harmonic reasons.

Arguments by composers, performers and others concerning tempered versus untempered intonation are presented, followed by an analysis of this writer's observations in preparing a twelve-tone work for performance, in which an aurally satisfying rendition of the music was found 1) to exhibit pitch variations consistent with Pythagorean and, in appropriate instances, just intonation; and 2) to indicate an aural process of enharmonic selection based on the acoustical relationships among notes and not necessarily corresponding to the enharmonic choices notated, perhaps arbitrarily, by the composer.

Noting that pitch adjustments for harmonic reasons have persisted despite traditional notation's inability to express them, the writer proposes that the twelve-tone composer's decision to relinquish, in effect, the available written means of distinguishing enharmonic pitches need not be interpreted to mean that such distinctions must cease to exist in performance.

Appendixes to this paper include a coordinated diagram of the systems of Pythagorean and just intonation in comparison with equal temperament; a description of the enharmonium, an electronic keyboard instrument designed and constructed to display the effects of untempered intonation; and a table of intervals.

ACKNOWLEDGEMENTS

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CONTENTS

INTONATION VARIABLES IN THE PERFORMANCE OF TWELVE-TONE MUSIC

By Michael Kimber

The consideration of intonation in twelve-tone music is prompted by an inconsistency between compositional theory and performance practice. While the tone material of twelve-tone music is, by definition, derived from the division of the octave into twelve equal semitones corresponding to the system of equal temperament[1]—the basis of present-day tuning of keyboard instruments—studies of the intonation of string players have revealed that equal temperament is *not* the pattern adhered to in actual performance.[2]

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[1] According to Joseph Rufer, in *Composition with Twelve Tones*, trans. by Humphrey Searle (London: Barrie and Jenkins, 1970), p. 100, "every twelve-note series consists of the twelve different notes of our tempered system." George Perle, in *Serial Composition and Atonality* (2nd ed.; Berkeley: University of California Press, 1968), p. 4, states that "the twelve notes of the set derive from a division of the octave into twelve equal parts." Milton Babbitt, in "Twelve-Tone Invariants as Compositional Determinants," *Problems of Modern Music*, ed. by Paul Henry Lang (New York: W. W. Norton, 1962), p. 109, describes his twelve "pitch classes" as "corresponding to the chromatically equal-tempered quantization of the frequency continuum."

[2] See Paul C. Green, "Violin Intonation," *Journal of the Acoustical Society of America*, IX (1937), 43–44; James F. Nickerson, "Comparison of Performances of the Same Melody in Solo and Ensemble with Reference to Equal Tempered, Just, and Pythagorean Intonations," *JASA*, XXI (1949), 462; idem, "Intonation of Solo and Ensemble Performance of the Same Melody," *JASA*, XXI (1949), 593; Charles Shackford, "Some Aspects of Perception, I: Sizes of Harmonic Intervals in Performance," *Journal of Music Theory*, 5 (1961), 162–202; idem, "Some Aspects of Perception, II: Interval Sizes and Tonal Dynamics in Performance," *JMT*, 6 (1962), 66–90.

Even though twelve-tone equal temperament on keyboard instruments considerably antedates the arrival, over fifty years ago, of twelve-tone music itself, the factors that have long determined intonation preferences apparently continue to override any supposed conditioning to equal-tempered tuning. Twelve-tone music, despite the multiplicity of enharmonic notation possibilities, is intended as a music having only twelve different "pitch classes"; yet the intonation scheme favored by string players, unconsciously to a great extent, involves between twenty and thirty audibly distinguishable pitches in an octave. Some present-day composers[3] have written music that intentionally exploits the resources of untempered intonation, but the "twelve-toners" have continued to exert a dominant influence on the contemporary idiom.

What are the implications of this inconsistency for the performance of twelve-tone music, not only by string players, but also by anyone whose intonation is not restricted as it is on a keyboard instrument? Does the fact that twelve fixed pitch classes do not adequately satisfy aural demands invalidate the technique of composing music with only twelve tones? Do we, or ought we, adapt our intonation preferences when performing twelve-tone music? If we are not using equal temperament, what pitches *are* we using in the performance of twelve-tone music or, for that matter, any music?

Intelligent consideration of such questions requires more knowledge than most musicians have of what might be called intonation theory. Because music education is strongly keyboard-oriented, musicians generally think of "in tune" in terms of a piano's tuning. Any intentional deviations from the equal-tempered scale are typically understood only vaguely, at best being justified as somehow musically desirable. The most familiar

[3] Lou Harrison, Benjamin Johnston, and Harry Partch, to name a few.

example of such an acknowledged musically desirable deviation is the high leading tone; it is frequently mentioned, yet without the slightest suspicion arising that this phenomenon might be only a more obvious manifestation of an overall pattern of departures from tempered tuning.

Equal temperament is further taken for granted in the teaching of music theory: the circle of fifths, in which enharmonic change is no more than a matter of spelling convenience, is a circle of tempered, not true fifths; and the elements of melody and harmony are assumed to be identical when, in fact, they are not always so. Numerous examples could be given attesting to the failure to teach music in terms other than those of equal temperament—or, as Harry Partch worded it, "the iniquitous determination of music education to withhold from students any adequate comprehension of the problems of intonation."[4]

While the subject of intonation can be approached intellectually through the calculation of vibration ratios, and the theoretical derivation of intervals, chords, scales, etc., the effects must really be heard to be appreciated and genuinely understood. The basic phenomena of untempered intonation can be demonstrated on a bowed string instrument, and such demonstrations will suffice to some extent. Anyone who is really serious about such matters may wish to try re-tuning a keyboard instrument in various ways.[5] The present writer has employed a specially constructed electronic organ that provides enharmonic choices for

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[4] Harry Partch, *Genesis of a Music* (Madison: University of Wisconsin Press, 1949), p. 297.
[5] The E. F. Johnson Company of Waseca, Minn., manufactures the Johnson Intonation Guide and Trainer, a three-octave organ equipped with knobs to vary the pitch of each note, and a second rank of pitches tuned to equal temperament for comparison. See John W. Travis, *Let's Tune Up* (Baltimore: Port City Press, 1968), pp. 150–151.

each degree of the chromatic scale. With this "enharmonium" it is possible to achieve a wide variety of reliable and accurate demonstrations and comparisons of untempered intonation.[6]

Measurements of String Intonation

Among the earliest scientific measurements of the intonation of string players were those made in the 1860s by the celebrated French physicist Alfred Cornu in collaboration with E. Mercadier. They concluded from their measurements of the intonation of professional string players that musical intervals belong to at least two distinct systems of different values: 1) melodic intervals, that is, intervals formed by tones that are played *consecutively*, agreeing with the intervals of the *Pythagorean* scale; and 2) harmonic intervals, whose tones are played *simultaneously*, being tuned according to the simple ratios of *just* (also known as "pure" or "natural") intonation.[7] The similarities and differences between these two systems, and their departures from equal temperament, will be described in the following pages.

The findings of Cornu and Mercadier, particularly concerning Pythagorean intonation, have been confirmed by modern investigation. Paul C. Green noted that violinists agree in the tendency to play Pythagorean rather than either tempered or just melodic intervals.[8] James F. Nickerson carried the investigation an important step further to find that Pythagorean intonation prevails not only in *solo* melodic playing, but in *ensemble*

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[6] Many of the musical examples in this paper were illustrated with the enharmonium in the course of the present writer's lecture-recital, "Intonation Problems in Twelve-Tone Music," School of Music, The Catholic University of America, October 22, 1974. (For a detailed description of the enharmonium, see Appendix II of this paper.)
[7] See Additions by the Translator, in Hermann Helmholtz, *On the Sensations of Tone*, trans. by Alexander Ellis (New York: Dover Publications, 1954), pp. 325 and 486–488, citing studies by Cornu and Mercadier, among them "Sur les intervalles musicaux," *Comptes Rendus de l'Académie des Sciences de Paris*, t. LXVIII (February 1869), 301 and 424.
[8] Green, "Violin Intonation," *JASA*, IX, 43–44.

performance of the same melody as well. Nickerson observed that "factors causing this pattern of intonation appear dominant over both ensemble (harmonic) demands and the often assumed 'cultural conditioning' from the equi-tempered intonation."[9] Charles Shackford directed his study specifically to measurement of the sizes of harmonic intervals in ensemble performance, and he found that here, too, Pythagorean values appear to be most representative of actual practice, thirds and tenths beginning to approach their just or natural values only when long-held.[10]

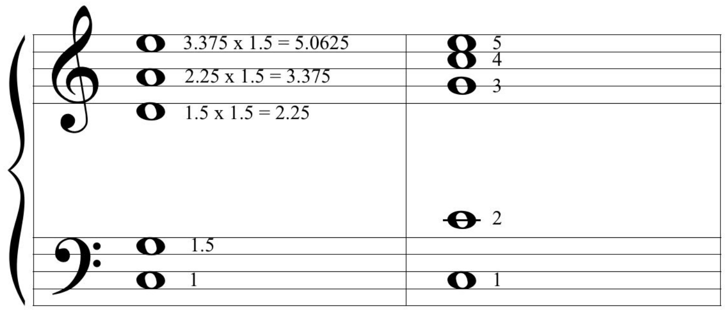
Pythagorean and Just Intonation Contrasted with Equal Temperament

Two systems of intonation have been mentioned in these studies: Pythagorean and just. The Pythagorean system is based on the true or pure—not tempered—perfect fifth (ratio 3/2, or 1.5). All notes in this system are generated either by pure fifths, or by their inversion, pure fourths (ratio 4/3, or 1.333…). The just system is based partly on pure fifths, but also on pure thirds, the basic unit being the pure major triad (4/5/6), whose tones correspond to the 4th, 5th and 6th partials of the harmonic series. Both systems—in fact, all systems of tuning in western music—assume the identity of the octave, 2/1.

The essential difference between Pythagorean and just intonation is in the contrasting sizes of the Pythagorean and just (pure) thirds. The Pythagorean major third derives from four pure fifths, which produce the ratio 1.5^4, or 5.0625; the just major third corresponds to the 5th harmonic partial, whose ratio to the fundamental or 1st partial is exactly 5.0. As a result, the just major third is a slightly smaller interval. (See Example 1.)

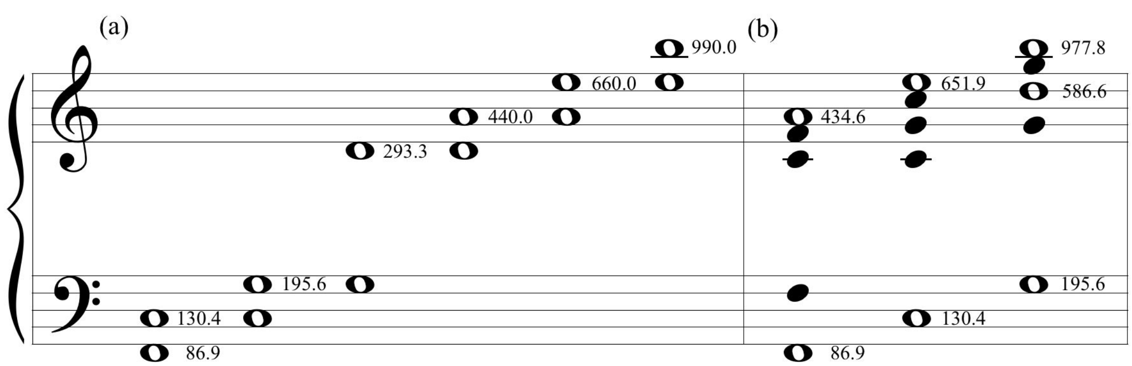
[9] Nickerson, "Intonation," *JASA*, XXI, 593.
[10] Shackford, "Perception, I," *JMT*, 5, 162–202.

Example 1. Pythagorean E ≠ just E



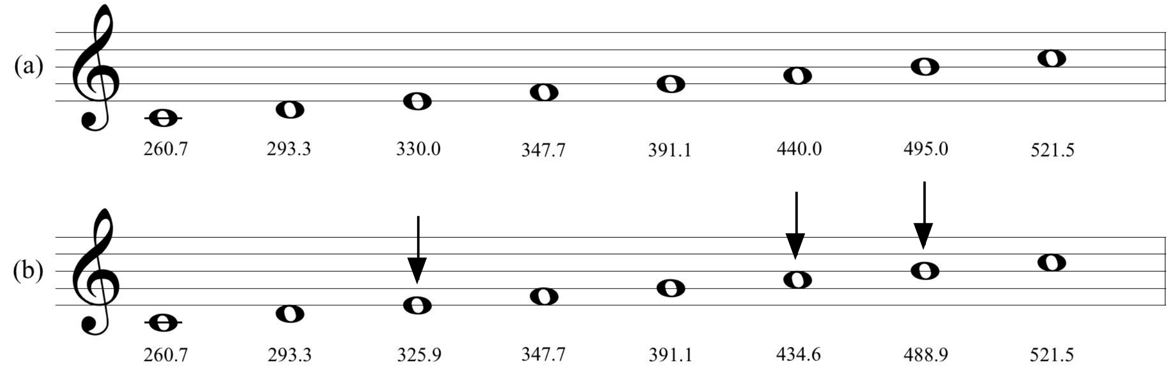
Example 2 illustrates the derivation of seven diatonic tones by both Pythagorean and just processes. Actual frequencies in Hertz (cycles per second or cps) have been given to enable pitch comparisons.

Example 2. (a) Pythagorean derivation; (b) just derivation



All these tones may be transposed to the same octave and arranged in order of increasing frequency to form the Pythagorean and just major diatonic scales. (Example 3.)

Example 3. (a) Pythagorean major scale; (b) just major scale



The third, sixth and seventh scale degrees (shown by arrows in Example 3b) are lower in the just scale than in the Pythagorean scale. These notes are the pure major thirds of the tonic, subdominant and dominant triads respectively. The amount by which each of these notes differs from its Pythagorean counterpart is known as a *syntonic comma*, and is slightly more than one-fifth the size of an equal-tempered semitone.

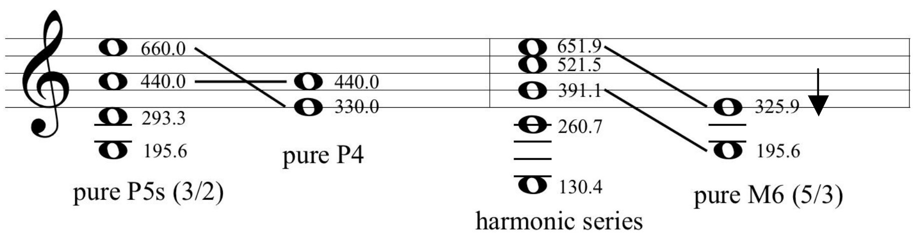
Tones of the just scale may be combined to produce pleasant-sounding harmonic progressions, especially of major triads. The individual tones sounded consecutively, however, have a less satisfying effect than the tones of the Pythagorean scale, due to the presence, in the just scale, of smaller whole tones between scale degrees 2–3 and 5–6, and larger semitones.

Even though the findings of Nickerson and Shackford indicate that players in ensemble seldom tune harmonically, string players still tend to adjust their double stops to the simple just ratios, meaning that in a double-stop third, sixth or tenth, one of the notes in the interval will depart a comma from its Pythagorean value.[11] String players can easily

[11] Christine Heman, *Intonation auf Streichinstrumenten* (Basel: Bärenreiter-Verlag, 1964), pp. 9–11, 38.

observe this phenomenon: the E that is a perfect fourth below the open A string is not the same pitch as the E that forms an agreeable double-stop major sixth above the open G string. The former E is the Pythagorean, agreeing with tuning by pure perfect fifths; the latter is just, lower by a syntonic comma and corresponding to the interval between the 3rd and 5th harmonic partials.

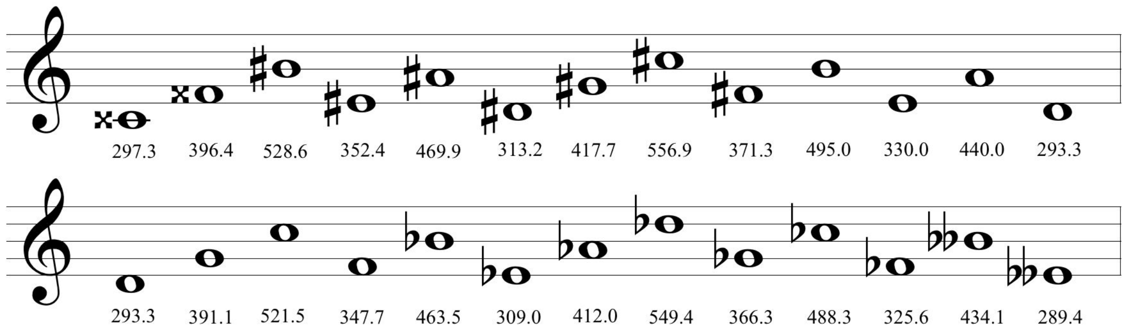
Example 4. Adjustment of E by a syntonic comma



In spite of this obvious phenomenon, many string players perpetuate the faulty practice of checking the intonation of melodic thirds, sixths and tenths by tuning them as harmonic intervals—either double stops or long-sustained chords in ensemble—and occasionally some wonder why the results are not always totally agreeable. The syntonic comma is the reason why.

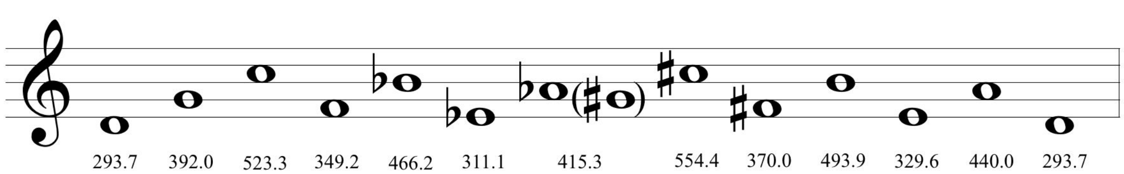
The Pythagorean series of pure fifths in Example 2 could be extended beyond these seven notes to generate both sharps and flats for any tonality. Example 5 shows twenty-five notes derived in this way and transposed into one octave, with frequencies in Hertz. Comparing the frequency of each note with that of its enharmonic note directly above or below, it is obvious that the two are not identical in pitch. This enharmonic pitch difference, known as a *Pythagorean comma*, slightly smaller than one-fourth of a tempered semitone, exceeds in size the syntonic comma by about 1/51st of a tempered semitone.

Example 5. Twenty-five tones in Pythagorean intonation



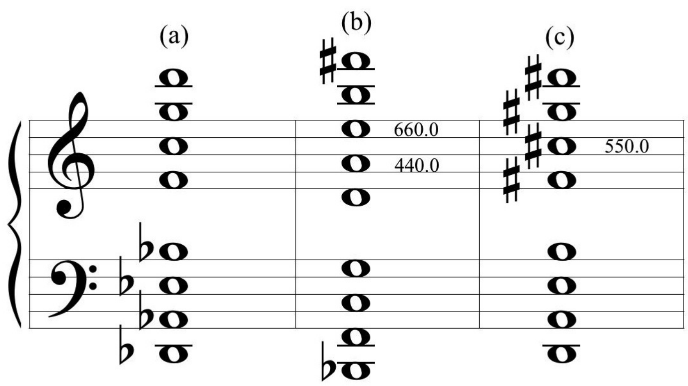
The Pythagorean enharmonic pitch difference, or comma, is generated by twelve pure fifths; twelve fifths exceed seven octaves by a Pythagorean comma ($1.5^{12} = 129.74632$, whereas $2^7 = 128.0$). In equal temperament, such enharmonic pitch differences are eliminated by reducing the size of each fifth by 1/12 of the Pythagorean comma. The resulting temperament is most practical for tuning instruments having provision for only twelve fixed pitches in the octave, but it is achieved at the expense of altering every interval from its Pythagorean value, some intervals more than others (and pure thirds and sixths even more), depending on the number of fifths required to generate the interval. For comparison with the Pythagorean pitches in Example 5, see the corresponding notes in equal temperament, with frequencies, in Example 6.

Example 6. The twelve tones of equal temperament



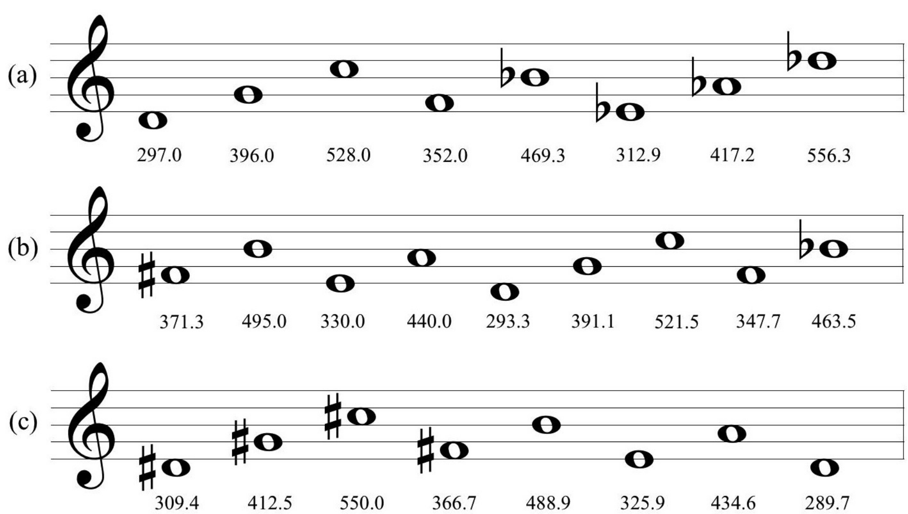
The system of just intonation can also be expanded to include sharps and flats by tuning pure major and minor thirds to the notes in a series of pure fifths (Example 7). Comparison of frequencies of these notes transposed into one octave (Example 8) with notes of Pythagorean intonation (Example 5) reveals an interesting coincidence: the notes that are tuned as pure thirds agree quite closely with their *enharmonic* counterparts in Pythagorean intonation. This is due to the near-equivalence of the Pythagorean and syntonic commas.

Example 7. Derivation of twenty-five tones in just tuning



Note: The frequencies of A 440, C# 550 and E 660 clearly display the relation 4/5/6 of the just major triad.

Example 8. Twenty-five tones in just intonation (see also Appendix I, p. 31)



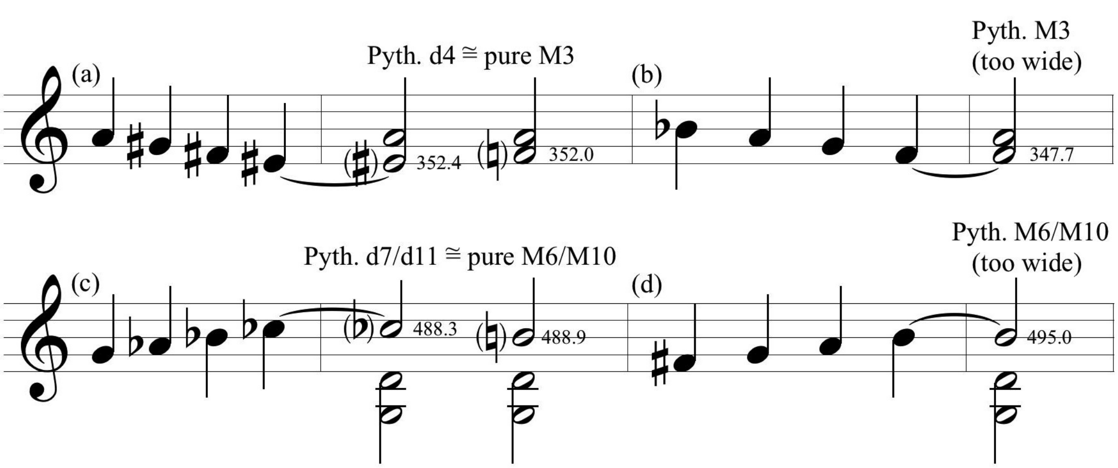
(a) tuned as pure thirds to notes in row (b)

(c) tuned as pure thirds to notes in row (b)

This near-coincidence of Pythagorean/just enharmonic notes can be used to demonstrate the existence of Pythagorean enharmonic pitch differences in string playing. It has already been noted that a melodic third or sixth does not form a satisfactory double-stop interval; it must be adjusted by a syntonic comma to sound in tune. If a note is adjusted instead by a Pythagorean comma, the effect will be practically the same. For example, as we have seen, E 330 is too high to form a pure major sixth above open G. Lowering E by a syntonic comma to 325.9 produces the pure interval. But F*b* 325.6 (a Pythagorean comma lower than E 330) would be hardly discernible from the actual pure major sixth above G.

Example 9 illustrates two double-stop tests for Pythagorean enharmonic pitch distinctions, based on the near-equivalence of the Pythagorean diminished fourth and diminished seventh with the just major third and major sixth respectively. The scale fragments shown may be thought of, and played, as extracts from the scales of (a) F sharp minor, (b) B flat major, (c) A flat minor, and (d) G major.

Example 9. Pythagorean enharmonic substitutions for pure thirds, sixths and tenths



The preceding, while not intended as an exhaustive treatment of the subject of intonation, has included reference to the essential points that should be recognized in order to follow the discussion of intonation in twelve-tone music.[12] To recapitulate briefly:

Pythagorean tuning, which has been found to be the pattern for melodic intonation and to prevail even over ensemble harmonic demands in most instances, is based on the pure, not tempered, fifth, and possesses enharmonic pitch differences—sharps being higher, and flats lower, than their enharmonics—by a Pythagorean comma, which is between one-fourth and one-fifth the size of a tempered semitone.

Just intonation, which influences the tuning of vertical (harmonic) intervals—especially in the case of double stops—involves not only pure fifths but also pure thirds, which differ from Pythagorean thirds by a syntonic comma, roughly one-fifth the size of a tempered semitone.

A near-equivalence of the most commonly used notes of just intonation with their enharmonics in Pythagorean intonation—due to the very small difference in size between the syntonic and Pythagorean commas—reduces to between twenty and thirty the number of *significantly* different pitches under consideration.

No interval in equal temperament (except the octave) agrees exactly with any interval in either Pythagorean or just intonation. While equal temperament is ordinarily regarded as the division of the octave into twelve equal intervals, its origin is the Pythagorean system, altered to eliminate enharmonic pitch differences. Tempered intervals, however necessary in tuning keyboard instruments, and however much musicians may be accustomed to hearing

[12] More detailed information, including diagrams and a table of Pythagorean, just, and equal-tempered intervals, will be found in the Appendixes to this paper.

them, must aurally be less than totally satisfactory, since performers consistently depart from them to follow patterns established by untempered fifths and thirds.

Proponents of Equal-Tempered Intonation

Considerations of the intonation of twelve-tone music, while not unprecedented, have in the past always been founded on the mistaken notion that the issue was between equal temperament and *just* intonation, rather than essentially between equal temperament and Pythagorean. Nevertheless it is possible to identify, on the one side, those who expected adherence to the tempered scale, and on the other side, those who maintained that such adherence, in twelve-tone or in any other music, does not occur.

Arnold Schoenberg called the tempered system a necessary "reduction of natural circumstances to *manageable* ones,"[13] and has been further quoted as saying, "Twelve-tone equal temperament is practical. There is no other popular medium available to the composer today."[14] In response to questions about intonation in his own music, he once replied that whenever he had occasion to take up the subject of intonation with string players, he always insisted on its tempered form ("ich habe immer *temperierte* Intonation verlangt"); furthermore, it was his opinion that the musical ear that has not assimilated the tempered scale is not sufficiently cultivated. "A singer who produces natural pitches is unmusical," he wrote, "just as one choosing to act on a street in a 'natural' way would be considered indecent."[15]

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[13] Arnold Schoenberg, *Theory of Harmony*, trans. by Robert D. W. Adams (New York: Philosophical Library, 1948), p. 24.
[14] Llewellyn S. Lloyd and Hugh Boyle, *Intervals, Scales and Temperament* (London: Macdonald, 1963), pp. 77–78.
[15] Joseph Yasser, "A Letter from Arnold Schoenberg," *Journal of the American Musicological Society*, 6 (1953), 53–62.

Others have expressed agreement with Schoenberg, specifically extending his demand for tempered intonation to tonal music as well. Rudolph Kolisch's remarks typify this attitude:

> …it is wrong to perform a piece of music…which owes its construction to the possibilities of temperament…and this includes practically every piece of importance since about 1700—in intervals other than tempered ones.… It is a well-known fact that especially string players are backward in this respect.[16]

J. Murray Barbour notes that apparently nobody is able to perform correctly in equal temperament, and that probably everybody is playing and singing out of tune most of the time. He claims, however, that

> their errors are errors from equal temperament. No well-informed person today would suggest that these errors consistently resemble departures from just intonation or from any other tuning system described in these pages. [Barbour describes Pythagorean tuning in detail; see especially pages 1 and 90 of his book.] Equal temperament does remain the standard, however imperfect the actual accomplishment may be.[17]

Barbour apparently was not familiar with, or did not take seriously, the investigations of Green and Nickerson, which had been published for some time prior to the appearance of his work.

The Comma Shift of Pitch in Just Intonation

Arguments in favor of equal temperament have also been advanced on the grounds that the comma makes performance in just intonation impractical. We are told that

> chromatic compositions cannot be played by way of pure fifths and thirds without inviting disaster. Either the general pitch level will rise or fall or eventually a pitch adjustment will have to be made, of a more or less radical

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[16] Rudolph Kolisch, "Über die Krise der Streicher," *Darmstädter Beiträge zur neuen Musik* (1958); quotation reprinted in a footnote to Gottfried Michael König's "Commentary," trans. by Ruth König, *die Reihe (Retrospective)*, 8 (English ed., 1968), 89–90.

[17] J. Murray Barbour, *Tuning and Temperament* (East Lansing: Michigan State College Press, 1953), p. 201.

> nature.… In performance…adjustments are preferentially made continually and minutely; thus they remain in accord with the principles of equal temperament.[18]

The writer goes on to quote Barbour's thesis that "equal temperament does remain the standard, however imperfect the actual accomplishment."

It has already been noted that the real issue is between equal temperament and Pythagorean—not just—intonation; nonetheless, one should understand what is meant by this rise or fall of the general pitch level with just intonation, if only to understand that this problem does not depend on equal temperament for its solution—Pythagorean intonation would also eliminate the problem in question—and that comma adjustments (either syntonic or Pythagorean), far from being pitch alterations of a "radical nature," are disturbing only when avoided.

If a progression of harmonies such as that in Example 10 is played using pure fifths and pure thirds in each chord, and without altering the pitch of any common tone from chord to chord, the final chord will be pitched a syntonic comma lower than the initial chord. This is simply because the pure third of the subdominant triad is a comma lower than the pure fifth of the supertonic triad (or, in this example, the dominant of the dominant). It is easy to assume that the pitch of a common tone should not be raised by a comma; however, doing so brings the final chord up to pitch, as shown in Example 11.

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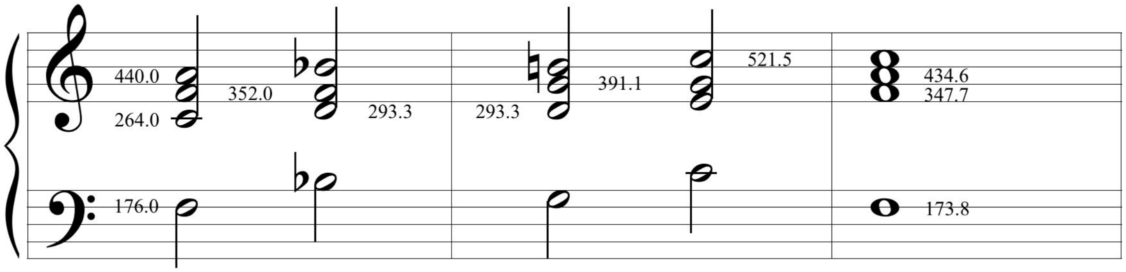
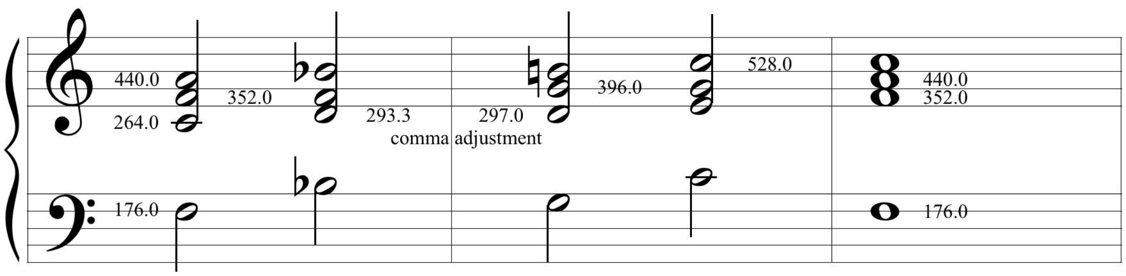
[18] William J. Mitchell, "The Study of Chromaticism," *Journal of Music Theory*, 6 (1962), 22.

Example 10. Just harmonic progression, illustrating fall in pitch level



Example 11. Just harmonic progression, with comma adjustment to prevent fall in pitch level



Aural comparison of these two possibilities—without comma adjustment (ending flat) and with adjustment (ending on pitch)—is likely to convince the listener not only of the superior effect of the latter, but also of the perfect acceptability of such comma adjustments.

The supposition that just intonation would be the only conceivable alternative to playing in equal temperament has led to a tendency to opt for the latter, the possibility of Pythagorean intonation never entering into consideration at all. Charles Rosen cites the opening of the third movement of Beethoven's String Quartet, Opus 130, as an argument against just intonation, noting that the tendency he has observed in this instance—to play A higher than B*bb*—is the reverse of what should occur in just intonation, but failing to see that this tendency is exactly in accord with *Pythagorean* intonation while having, of course, nothing at all to do with equal temperament. Rosen argues further against the "dangers of theory applied to performance" by attributing this theory that string players should use just intonation to the famous 19th-century violinist Joseph Joachim who, Bernard Shaw savagely claimed, did not play in just intonation, but simply out of tune.[19] (If Joachim really did use just intonation in a melodic way, he may very well indeed have sounded out of tune.)

Proponents of Untempered Intonation

There are, on the other side, those who feel the inadequacies of equal temperament as a basis for intonation and realize that musically satisfying intonation differs from it in certain ways, even though they may not always fully understand the theoretical basis for this. Of particular interest, in the light of Schoenberg's statement that he always insisted on tempered intonation, is the observation by Raphael Hillyer, who as violist of the Juilliard String Quartet played in rehearsals of Schoenberg's music attended by the composer himself. In these rehearsals Schoenberg never mentioned the subject of intonation, even though, Mr. Hillyer is convinced, the quartet did not temper their intonation, nor did they feel especially inclined to do so on grounds that the music was twelve-tone.[20]

Paul Hindemith, an accomplished performer on viola and numerous other instruments, as well as composer and theorist, observed that all musicians except keyboard players use untempered intonation and are continually faced with the problem of "disposing of the comma." He attributed intonation problems in extremely chromatic music to the increased difficulty of handling the comma.[21] Critical of the tacit acceptance of equal temperament, he described the system as "a compromise which is presented to us by the

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[19] Charles Rosen, *The Classical Style* (New York: W. W. Norton, 1972), pp. 27–28
[20] Raphael Hillyer, in a conversation with the present writer on January 30, 1974.
[21] Paul Hindemith, *Craft of Musical Composition, Book I*, trans. by Arthur Mendel (4th ed.; New York: Associated Music Publishers, 1945), pp. 44–45.

keyboard as an aid to mastering the tonal world" which "then pretends to be that world itself." The atonal style he regarded as "the uncritical idolatry of tempered tuning."[22]

Composer Lou Harrison has noted that

> voices, slide trombones, and strings do not play equal temperament well; they tend to introduce good intervals, and what's more play 22–25 or more pitches. Thus a "twelve-tone" work for voices or these other instruments is an anomaly, ill-founded and, it will, of course, be "adjusted".[23]

Other composers, while by no means endorsing untempered intonation, have made statements that cast doubt on the ability of the tempered scale to dictate actual intonation. Walter Piston has admitted that "performers as well as listeners lend tonal inflexions to music whose intended scale is that of equal temperament," calling this one of the "facts of musical communication" which "should be faced by composers, performers, and listeners."[24] Even George Perle, after stating that "in twelve-tone music there is, in principle at least, no difference in the meaning of enharmonically equivalent notes," concedes that "in much of this music certain 'voice-leading' or harmonic implications seem to be suggested by the preference for one rather than another spelling in given instances."[25] If composers are influenced in this way, it is certainly not implausible that performers would allow themselves corresponding freedom in the intonation of pitch classes.

One might still argue that twelve-tone music demands its own special adaptation of intonation technique, noting that none of the aforementioned intonation studies made use of twelve-tone material—although Shackford's material was fairly freely chromatic. Such an

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[22] Ibid., p. 155.
[23] Lou Harrison, *Lou Harrison's Music Primer: Various Items about Music to 1970* (New York: C. F. Peters, 1971), p. 16.
[24] Walter Piston, "More Views on Serialism," *The Score*, 23 (July, 1958), 48.
[25] George Perle, *Serial Composition and Atonality* (2nd ed.; Berkeley: University of California Press, 1968), pp. 3–4.

assertion raises the question of the audibility of twelve-tone technique itself—whether it is possible, in the hearing of a work, to detect serial process in the organization of pitch. A series of articles appearing in *The Score* during 1958 was devoted to just this question. In discussing it, both Roger Sessions and Roberto Gerhard emphasized that the sole importance of serialism is as part of the composer's technique—that its function is to govern the composer's choice of materials—and that it is not meant to be evident and cannot, in fact, be expected to be audible. It was noted that Webern resented performers' inquiries into his compositional technique and was adamant that one "should know how the work should be played, not how it was made."[26] Judging from their remarks, these composers certainly would not be in favor of seeking out serial technique *per se* in order to justify the use of tempered intervals in performance.

Detecting and Analyzing Enharmonic Choices in Twelve-Tone Music

The opinions of composers and others cited above, while representing the sides of the issue, do not resolve the question of intonation in twelve-tone music one way or the other. It is logical, however, that if an aurally satisfying rendition of twelve-tone material were to reveal instances of enharmonic choice as evident in the actual pitches of notes, this would at least establish untempered intonation as a viable alternative alongside the theoretical dictum in favor of tempered intonation in the performance of serial music.

Detection of enharmonic choices cannot ordinarily depend on a conscious awareness of the comma effect, since most musicians have never been trained to recognize this as such. However, double-stop tests such as those illustrated in Example 9 provide a systematic

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[26] See Roger Sessions, "To the Editor," *The Score*, 23 (July 1958), 58–64; and Roberto Gerhard, "Apropos Mr. Stadlen," op. cit., 50–57; responding to the article by Peter Stadlen, "Serialism Reconsidered," *The Score, 22* (February, 1958), 12–27.

method of identifying Pythagorean enharmonic pitch variables. The present writer, equipped with this simple but effective method, began to discover early in the study of Riccardo Malipiero's *Ciaccona di Davide*, a twelve-tone work for viola and piano composed in 1970, that the same note did not always have the same pitch, but even more importantly, that this pitch variability was consistent, not random or haphazard. For example, at certain places in the music pitches identifiable as Pythagorean E, F, B*b* and B occurred, while in other specific instances their less likely Pythagorean enharmonics, F*b*, E#, A# and C*b*, could be consistently and positively identified. It became apparent that in spite of the composer's choices of notation, pitches must be aligning themselves to form coherent, untempered intervallic patterns with surrounding pitches. This phenomenon brings to mind the acoustical physicist Arthur Benade's criticism that composers of twelve-tone music "are no better in their disregard of the physical world [of sound] than are sculptors in wood who try to ignore effects of the grain on the texture, working and strength of their carvings."[27] Obviously the factors that imply relationships among tones persist even if ignored.

The intonation choices in the *Ciaccona* were later found to correspond in practically every instance to pitches on the enharmonium, an instrument that was tuned to represent a reasonably broad spectrum of both Pythagorean and just intonation with considerable accuracy. (See Appendix II.) Such experience contradicts Barbour's contention that intonation consists of nothing more specific than mere "errors from equal temperament" and the premise that modulatory freedom necessitates an equal division of the octave.

This experience also demands an explanation of the preference for one enharmonic choice over another. It appears that there is an observable tendency for enharmonic

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[27] Arthur H. Benade, *Horns, Strings and Harmony* (New York: Doubleday, 1960), p. 89.

selections to be decided on the basis of proximity in a series of pure perfect fifths. In analyzing such decisions, two main considerations must be kept in mind: 1) individual tone connections and 2) contextual relationships. In most cases the first consideration is more important: the connection from one tone to the next is achieved by way of the shortest route along a series of fifths (e.g., a diatonic semitone, via five fifths, is preferred over a chromatic semitone, via seven fifths). Obviously there must be exceptions to this, however (a chromatic scale cannot consist entirely of diatonic semitones, for example); this is where the influence of contextual relationships is felt. Collections of intervals, rather than individual intervals, may have to be viewed as a whole. Pitch order can influence selection; in addition, the elements of metric and rhythmic emphasis, pitch duration, and melodic contour can direct attention toward certain intervals and away from others, establishing a hierarchy of relationships—intended or not—among notes of a pattern.

These observations have been deduced by analysis from the majority of cases encountered in the *Ciaccona*. It must be acknowledged, however, that the process of pitch organization by the auditory nervous system, not yet fully understood, is an area in which there is still much psychoacoustical work to be done.[28]

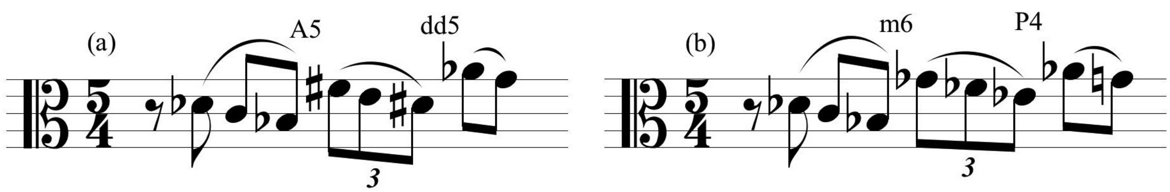
[28] For an insight into research of this type that has been done, see: Paul C. Boomsliter and Warren Creel, "The Long Pattern Hypothesis in Harmony and Hearing," *JMT*, 5 (1961), 2–31; idem, *Interim Report on the Project on Auditory Perception* (Albany: State University College, 1962); idem, "Extended Reference: An Unrecognized Dynamic in Melody," *JMT*, 7 (1963), 2–23.

Enharmonic Choices in Riccardo Malipiero's *Ciaccona di Davide*

In the following examples from the viola part of Malipiero's *Ciaccona di Davide*,[29] the music has been enharmonically re-notated as necessary to identify, in Pythagorean terms, the actual pitches selected during preparation of the work for performance. (Note: In Malipiero's own notation, the accidental is valid only for the note before which it stands.)

Example 12 illustrates intervals notated with a mixture of sharps and flats resulting in both an augmented fifth and a doubly diminished fifth. These were adjusted to minor sixth and perfect fourth respectively.

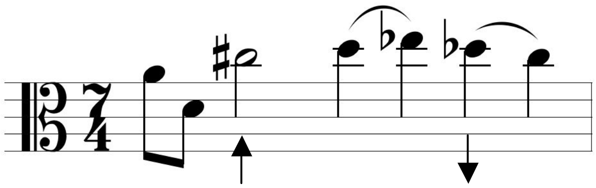
Example 12: 1 meas. before [3] : (a) written, (b) played



In the third measure after rehearsal number 3, the composer has written both C# and D*b*. Not intending an actual enharmonic distinction, his choice of notation was, as always, according to him, "only a question of reading: I try to facilitate the reading of my music (sufficiently difficult, I am told, in itself!)"[30] Nonetheless, he has here reflected an appropriate intonation choice: the C# ("leading" to D) is played consistently higher than the D*b* ("leading" to C)—diatonic semitones in each instance.

[29] Riccardo Malipiero, *Ciaccona di Davide* (Milan: Edizioni Suvini Zerboni, c1971). Examples quoted by permission of the publisher.
[30] Riccardo Malipiero, in a letter of February 24, 1974, to the present writer.

Example 13. 3 meas. after [3] : written and played



A complete quotation of the series on which the composition is based,[31] followed by its inversion transposed down a whole tone, happens to provide for comparison of hexachords having identical pitch content—in terms of equal temperament—except for the third note of each. (See Example 14.) The first six notes of the first measure quoted begin with C# and end with B; the first six notes of the following measure begin with B and end with D*b*. Also shared are the notes A# (B*b*), C and D, whose actual intonation was found to differ from one occurrence to the next, each of these three notes returning a comma higher in the next measure.

The decision of how to tune the tritone between the sixth and seventh notes of the first measure was crucial in determining the pitch level of the notes that follow. Diatonic

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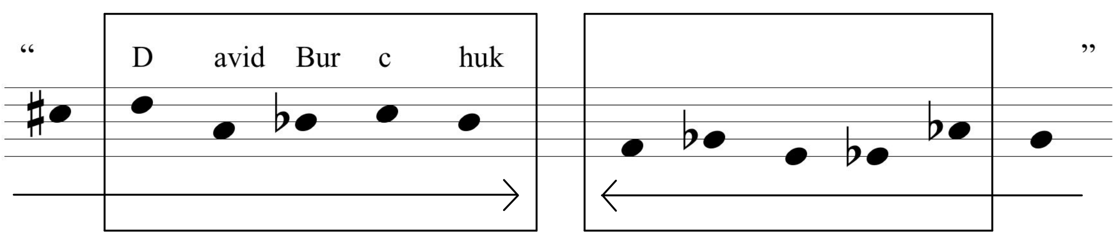
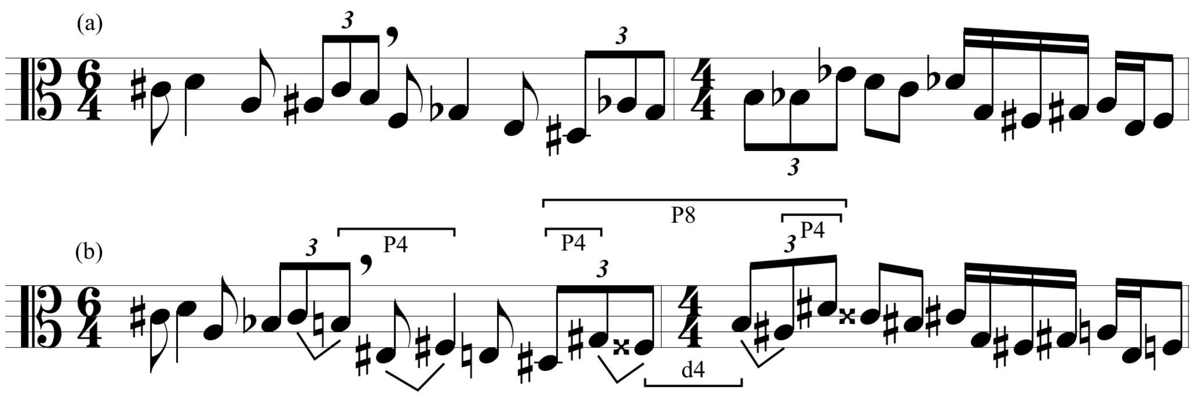
[31] Malipiero explains the derivation of the series in a prefatory note to the music: “This Ciaccona is dedicated to David Burchuk as a token of affectionate friendship and spiritual fraternity. The composition is based on a row not devoid of a sense of magic:



Present writer’s note added in 2011: Malipiero’s example is in treble clef. The “sense of magic” may be explained, as his arrows show, by the second hexachord being the retrograde of the first, transposed by a tritone, as well as by the row having been inspired by the name David Burchuk. A personal note: violist, teacher and conductor David Burchuk and his wife Rhoda were the founders in 1950 of Dale Music Company, Inc., a music store in Silver Spring, Maryland. Mr. Burchuk, who had also taught at Maryland State and Catholic University, passed away September 15, 1975, in Washington, D.C., at the age of 63.

semitones with adjacent notes were preferred to chromatic, and—possibly due to rhythmic emphasis—a descending pure fourth, B–F#, was preferred over a descending pure fifth, C–F. (In this context, using only diatonic semitones, it would be impossible to have both.)

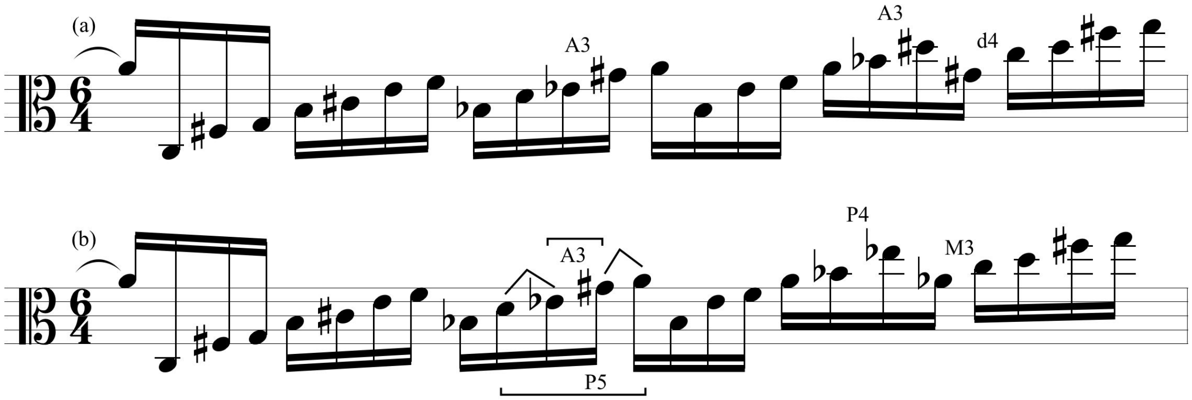
Example 14: Meas. 2–3 after [8]: (a) written, (b) played



Whatever the underlying reasons, the choices of the first measure were duplicated, in inversion, in the second measure. Interestingly, the diminished fourth across the bar-line apparently yielded the tendency to become a major third to stronger demands: preservation of the fourth-fifth-octave configuration (D#–G#–A#–D#) and the predilection for diatonic semitones.

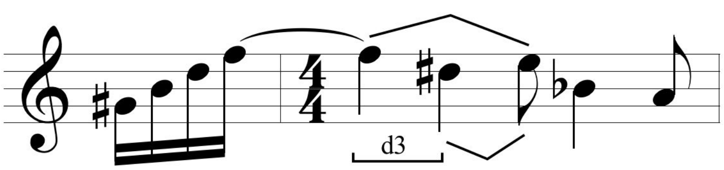
Similar to the case of the diminished fourth just mentioned is that of the first augmented third (E*b*–G#) in Example 15, which here seems justifiable inasmuch as it, unlike a perfect fourth, provides neighboring diatonic semitones to a perfect fifth. The same interval recurs between the notes B*b* and D#; this time, however, there is no rationale for the interval, and, in fact, the diminished fourth (G#–C) that follows—here also unjustifiable—is further argument for enharmonically flatting both D# and G# in this instance.

Example 15. 2 meas. after [25] : (a) written, (b) played



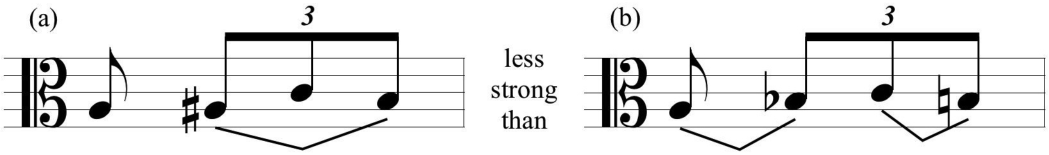
The interval of the Pythagorean diminished third, aurally indistinguishable from the inferior-sounding small whole tone of the just scale (see Example 2 and text on page 7), is nevertheless effective in the context of each component moving by diatonic semitone to a central pitch. (Example 16.)

Example 16. 1 meas. before [10] : written and played



The possibility of retaining the diminished third A#–C in Example 14 had been considered but seems there to have been ruled out by the stronger demand for two contiguous diatonic semitones, A–B*b* and C–B. (Example 17.)

Example 17. From 2 meas. after [8] : (a) written; (b) played



The notation as played in the next two examples will be meaningless without some explanation. Since pitch inflections by a syntonic comma cannot be expressed in the terms of traditional notation, but inflections by a Pythagorean comma can (by enharmonic respelling), just intervals will here be represented by the Pythagorean intervals most nearly the same in size, recalling the enharmonic double-stop test (Example 9, p. 11).[32]

In double-stop thirds or sixths, the pitch of one member will normally depart from its melodic (Pythagorean) value to form a pure interval with the other member. This change of pitch by a comma is, in effect, an enharmonic alteration. In the next example, the Pythagorean minor sixth, C#–A, would be too narrow harmonically; the C# must be lowered by a comma, making it the practical equivalent of D*b*. In the *major* sixth that follows, open C is a comma too low for the A already sounding. Unless one had begun by playing the first double stop entirely a comma lower, the pitch of A will have to change, for the pure major sixth above open C is about the pitch of Pythagorean B*bb*, not A. (Example 18.) This alteration of a common tone is not unlike that employed in the progression in just intonation to prevent a fall in general pitch level. (See Example 11 on p. 16.)

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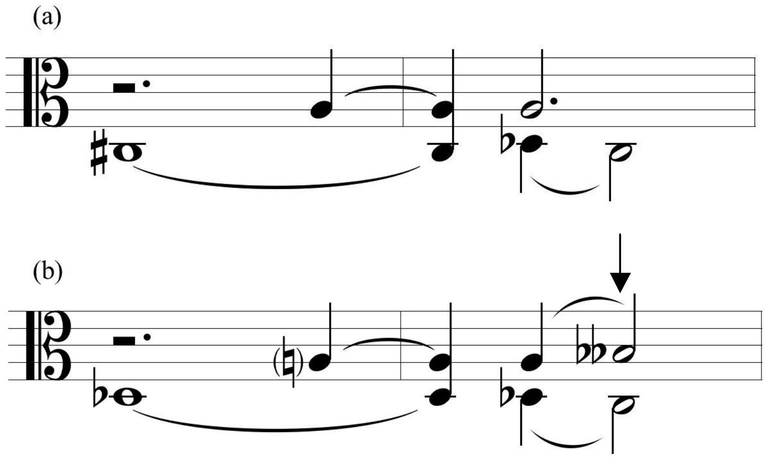
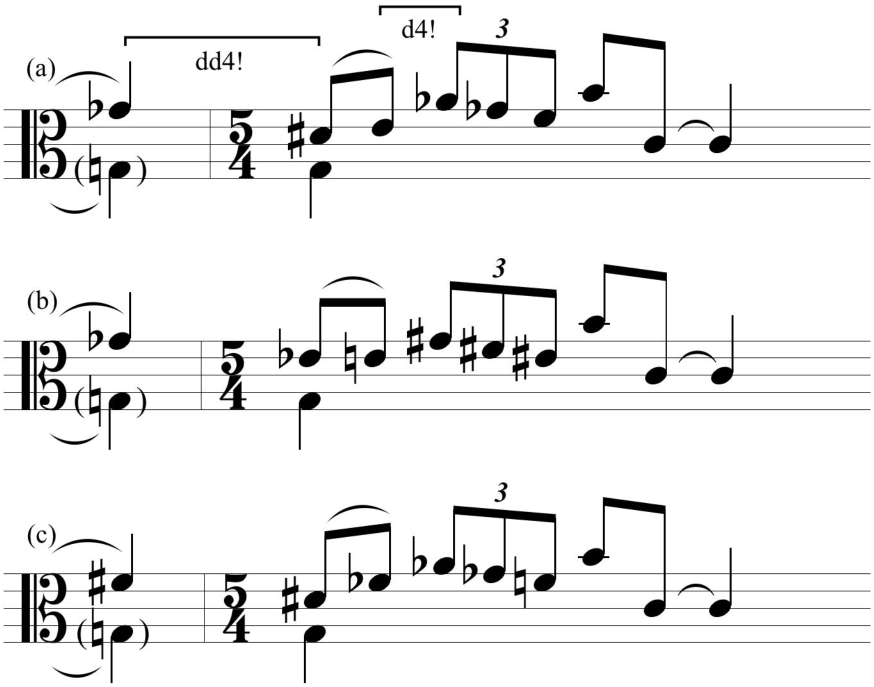
[32] Enharmonic respelling to represent just intervals in Pythagorean terms is used here for illustrative purposes only and is not advocated as a practical means of notating just intonation in music for performance.

Example 18. Meas. 2–3 after [2] : (a) written; (b) played



In Example 19(a), two implausible-looking melodic intervals appear: G*b*–D# and E–A*b*. A string player, adapting this passage to read in terms of string technique, would probably rewrite it as shown in Example 19(b). However, the actual rendition, expressed in Pythagorean terms in Example 19(c), preserves most of the composer's notation, with only two notes being respelled: G*b* becomes F#, the major seventh above open G; and E is lowered a comma (represented by F*b*) to form a pure major sixth above G. The D# is actually an appropriate notation in a sense, for Pythagorean E*b* would be a comma too low in the double stop. The melodic succession D#–F*b* looks, but does not sound, strange (it corresponds to the small chromatic semitone in just intonation); and the implausible-looking intervals mentioned earlier have disappeared.

Example 19. At 6 : (a) written; (b) notated idiomatically for string technique; (c) played



These examples represent realistic solutions to some characteristic problems encountered when dealing with the ambiguous notation of twelve-tone music. While a kind of musical intuition will probably always remain the most reliable guide in the refinement of intonation in twelve-tone (or any other) music, it can be positively helpful to be aware of the variables that exist, so as to be favorably equipped to handle them. This means first of all abandoning the notion that intonation will somehow be equal-tempered simply because the composer either did not conceive of, or was not aware of, or was not concerned about, or did not even believe in, enharmonic pitch variability.

The Influence of Piano Temperament on Intonation

One important matter still has to be considered, namely the influence of tempered piano tuning on intonation in accompanied performance. The studies of Nickerson and

Shackford make it clear that Pythagorean intonation prevails over harmonic demands in ensemble playing, indicating that the tuning of consecutive intervals is the primary desideratum. From this it is reasonable to assume that Pythagorean intonation will similarly prevail despite the implication that it should adjust to tones already sounding on the piano. Unison or octave passages may require special treatment; however, these are the exception in twelve-tone music.

Pablo Casals once advised a student that when playing sonatas with piano, "do not be afraid to be out of tune with the piano; it is the piano that is out of tune. The piano, with its tempered scale, is a compromise in intonation.[33]

The coexistence, in performance, of tempered tuning and untempered intonation has also been approved, interestingly enough, by the author of a book on electronic music, who noted that

> since the intervals of the scale are not exactly fixed by mechanical means, the intonation of a good string player is often more exact in terms of pitch than the corresponding note sounded on an instrument tuned to the equal temperament scale. This clash of pitch can frequently be heard during a violin solo with piano, and is in fact a point in favour of the instrument and not, as often thought, a fault in intonation; it is the piano which is wrong.[34]

There are, in addition, acoustical considerations: it should be noted that due to the transient nature of piano tone, its tuning is bound to have comparatively brief interaction with the intonation of sustained tones as produced on a bowed string instrument. The present writer has played the enharmonium with piano accompaniment, and the effect was perfectly acceptable. In fact, the attempt to dramatize differences between tempered and untempered

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[33] Samuel and Sada Applebaum, *The Way They Play* (Neptune City: Paganiniana Publications, 1972), p. 272.

[34] Alan L. M. Douglas, *The Electrical Production of Music* (London: Macdonald, 1957), p. 41.

tuning using piano and enharmonium is somewhat of a failure. On the other hand, the clash between the enharmonium and an *organ* in equal temperament—both instruments producing sustained tones—is strikingly and immediately evident.

Conclusion

While equal-tempered intonation is theoretically postulated for the performance of twelve-tone music, it should now be apparent that another, perhaps more realistic possibility also exists—that of treating twelve-tone music as embodying an extension of the enharmonic latitude that has been present in music since the beginnings of polyphony, when the interplay of melodic and harmonic forces began to require pitch variability. The fact that traditional musical notation never acquired the means of distinguishing pitches differing by a syntonic comma does not mean that such differences did not occur; likewise, the decision of twelve-tone composers to relinquish the available means of distinguishing pitches that differ by a Pythagorean comma as well is not to say that such pitch variations have ceased to exist, but that performers now simply have increased responsibility of enharmonic choice.

Addendum to the 2011 reprinting:

During the 37 years since this paper was written, many musicians have become more aware of tuning and intonation options, in part due to the early music movement's revival of unequal temperaments, in which some keys sound more harmonious—and others less so—than in today's equal temperament. With this has come an ear-opening realization that there exists no "ideal intonation" based on any inflexible set of pitches, equal-tempered or other, rather that there are choices to be made.

The precise pitch frequencies and cent values given in examples in the preceding text and in the tables of tunings and intervals in Appendix I and Appendix III are provided only for reference and comparison; they are not intended to advocate strict and unyielding adherence to one system or another. Beautiful intonation results from weaving an artistic path between the sometimes contrasting melodic and harmonic influences that incline us to inflect our intonation slightly above or below an equal-tempered "average." The point to remember is that attempting to realize a division of the octave into twelve equal semitones in performance is not the goal if we aspire to anything better than "average" intonation.

APPENDIX I

The Pythagorean and Just Intonation Systems

Pythagorean

cents*		Hz
522	C##	594.7
1020	F##	792.9
318	B#	528.6
816	E#	704.8
114	A#	469.9
612	D#	626.5
1110	G#	835.3
408	C#	556.9
906	F#	742.5
204	B	495.0
702	E	660.0
0	A	440.0
498	D	586.7
996	G	782.2
294	C	521.5
792	F	695.3
90	B*b*	463.5
588	E*b*	618.1
1086	A*b*	824.1
384	D*b*	549.4
882	G*b*	732.5
180	C*b*	488.3
678	F*b*	651.1
1176	B*bb*	868.2
474	E*bb*	578.8

Pythagorean

*above A 440 Hz

compare

Just (left)

520	D	594.0
1018	G	792.0
316	C	528.0
814	F	704.0
112	B*b*	469.3
610	E*b*	625.8
1108	A*b*	834.4
406	D*b*	556.3

Just (right)

590	D#	618.8
1088	G#	825.0
386	C#	550.0
884	F#	733.3
182	B	488.9
680	E	651.9
1178	A	869.1
476	D	579.4

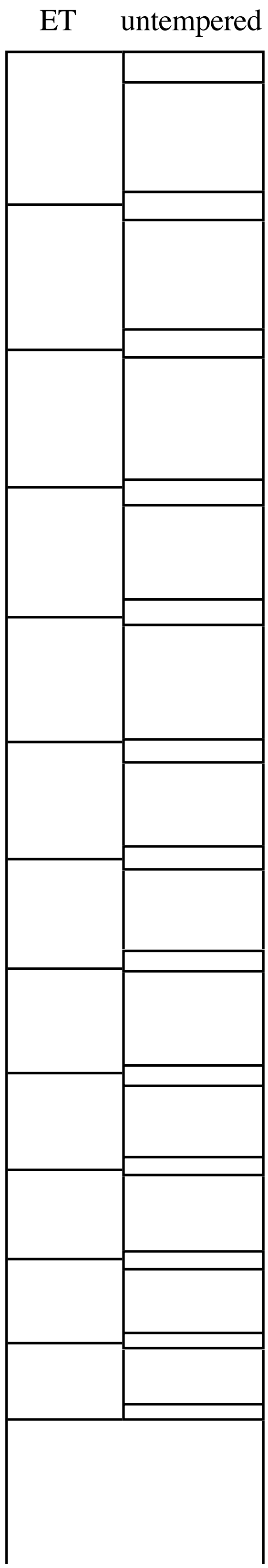
compare

Equal Temperament

cents*		Hz
0	A	440.0
100	A#/B*b*	466.2
200	B	493.9
300	C	523.3
400	C#/D*b*	554.4
500	D	587.3
600	D#/E*b*	622.3
700	E	659.3
800	F	698.5
900	F#/G*b*	740.0
1000	G	784.0
1100	G#/A*b*	830.6

To the left of each note name is the number of cents in the interval from A up to that note. (1 cent = 1/100 of an equal-tempered semitone; 1200 cents = one octave)

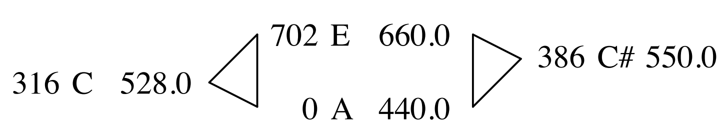
To the right of each note name is the frequency of that note in the octave above A 440 Hz.



The scale at left illustrates the actual spacing of notes in one octave on a viola string 15 inches (384 mm) long. The marks on the left are for the scale of equal temperament; those on the right are for twenty-five untempered pitches.

The diagram on the preceding page is a coordinated representation of the systems of Pythagorean and just intonation with the equal-tempered division of the octave presented below it for purposes of comparison. The central column shows the notes in a series of pure perfect fifths. The pure (not tempered) perfect fifth (ratio 3/2)—or its inversion, the pure perfect fourth (ratio 4/3)—is the interval that generates the notes of Pythagorean intonation.

The notes in the columns to the left and right are tuned as pure major thirds (ratio 5/4) and pure minor thirds (ratio 6/5) to adjacent notes in the central column. Pure thirds in combination with pure fifths generate the notes of just intonation. The diagram below shows sample configurations of pure minor and major triads:



The Pythagorean and just systems are not limited to those notes shown in the diagram. The column of pure fifths could extend indefinitely without repeating any of the same pitches exactly (although 53 pure fifths exceed 31 octaves by only about 3.615 cents). Additional columns tuned as pure thirds could also be added. The notes represented here are those most useful, however.

Twelve pure fifths exceed seven octaves by about 24 cents—the Pythagorean comma. This is the amount by which Pythagorean enharmonic notes differ, e.g., 90 B*b* and 114 A#.

The just or pure third differs from the Pythagorean (fifth-generated) third by about 22 cents—the syntonic comma. This is the amount by which notes in just intonation differ from Pythagorean notes of the same name, e.g., 386 C# and 408 C#.

Enharmonic notes in just intonation can differ by varying amounts. Those shown in the diagram differ by about 20 cents—the diaskhisma—e.g., 590 D# and 610 E*b*. Other differences also occur: three pure major thirds fall short of an octave by 41 cents, while four pure minor thirds exceed an octave by 63 cents, suggesting that the harmonic intonation of augmented triads and diminished seventh chords would need to be inflected or adjusted, *though not necessarily made to conform to equal temperament*.

Certain notes in just intonation and their enharmonic counterparts in Pythagorean intonation may be considered for practical purposes identical, the difference of only 2 cents—a skhisma—being nearly imperceptible. Compare the pitches in the left-hand column of the diagram on page 31 with those in the upper third of the central column; likewise, the pitches in the right-hand column with those in the lower third of the central column (see arrows).

Equal temperament may be considered the most practical attempt to represent the notes of both systems of intonation within the limitation of twelve inflexible pitches to the octave. Since twelve pure fifths exceed seven octaves by about 24 cents, each fifth must be tempered (that is, mistuned) by about 2 cents (a skhisma, but not precisely the same skhisma as encountered in untempered intonation) to eliminate enharmonic pitch differences. However, this mistuning, while slight in the case of a single fifth, accumulates with each successive fifth to make nearly all tempered intervals audibly distinguishable from untempered ones.

APPENDIX II

The Enharmonium

The enharmonium is an electronic organ designed for the purpose of enabling reliable demonstrations and instantaneous comparisons of enharmonic pitch differences in Pythagorean and just intonation. Its essential features are:

1) a tuning that provides enharmonic tones for each degree of the chromatic scale, and
2) a keyboard that exhibits a systematic and consistent arrangement of the intervals of the enharmonic chromatic scale.

I. Tuning

The enharmonium could be tuned entirely Pythagorean, i.e., to the notes in the vertical axis (central column) of the diagram in Appendix I, and represent fairly well the intervals of just intonation by enharmonic substitution, e.g., by using Pythagorean G*b* instead of F# for the pure third of the just D major chord. Yet the Pythagorean diminished fourth—smaller by only a skhisma[1] than the pure major third—beats fast enough to be noticeable in a sustained chord;[2] thus a strictly Pythagorean tuning was deemed less than wholly satisfactory.

On the other hand, the tuning could be entirely just, corresponding to the notes in the horizontal axis of the diagram in Appendix I. In this case, the fifths and thirds of quite a

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[1] The skhisma—about 1/51 of a tempered semitone—is the difference between the Pythagorean and syntonic commas and is also the difference between the syntonic comma and the diaskhisma.

[2] The Pythagorean diminished fourth D 293.33 Hz to G*b* 366.25 Hz beats about 1.65 times a second. Cf. Pythagorean D–F#, 18.33 beats/second; and equal-tempered D–F#, 11.65 beats/sec.

large number of triads would be absolutely pure, but there are flaws: two intervals—F#–D*b* and D#–B*b*—resemble equal-tempered fifths, each being a skhisma narrower than a pure fifth. This may seem a minor disadvantage, but it does interrupt an otherwise regular, Pythagorean series of pure fifths, and the "pure triads" F#–B*b*–D*b* and D#–F#–B*b* are less well in tune than if a purely Pythagorean tuning had been the basis.

In order to represent both systems of intonation satisfactorily, the skhisma must somehow be dispersed. Experimentation has shown that the most practical way of doing this is to tune so that the Pythagorean diminished seventh comes out as a pure major sixth—a skhisma wider—since beating in a mistuned major sixth is especially prominent.

Absolutely beatless major sixths are achieved by an adjustment of the fifths so slight that it must be considered inaudible. Nine pure fifths would produce a Pythagorean augmented second—the complement of the diminished seventh—plus five octaves, which is a skhisma wider than a pure minor third plus five octaves. Contracting each of the fifths theoretically by 1/9 of a skhisma (i.e., about 1/460 of a tempered semitone!) creates a tuning in which the Pythagorean augmented second and diminished seventh become identical to the pure minor third and major sixth respectively.

The resulting tuning, aurally indistinguishable from either Pythagorean or just intonation, is capable of accurately representing both. The beating of the enharmonium fifth is exceedingly slow[3] as is the beating of the enharmonium just major third.[4] The cumulative

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[3] The enharmonium fifth D 293.37 Hz to A 440 Hz beats only once every 9.07 seconds. Cf. the corresponding fifth in equal temperament, which beats once every 1.01 seconds.

[4] The enharmonium just major third D 293.37 Hz to F# 366.66 Hz beats only once every 5.44 seconds. The ratios for the enharmonium's just major triad—4.0005016 : 5 : 6—are exceedingly close to the ideal 4 : 5 : 6. Cf. the triad with a Pythagorean diminished fourth substituting for a pure major third: 4 : 4.9943612 : 6.

error in this tuning is not serious, as evidenced by the size of the enharmonium comma—somewhat smaller than either Pythagorean or syntonic, but larger than the diaskhisma.

In mathematical terms, the enharmonium fifth is contracted so that nine fifths, instead of generating the ratio 38.443359 (the ninth power of 1.5), produce exactly the ratio of a pure minor third plus five octaves—38.4 (that is, 1.2 x 2^5). Such a fifth has the ratio 1.499811926 (the ninth root of 38.4) instead of exactly 1.5.

The actual tuning procedure begins by tuning nine pure fifths, then lowering the last note as necessary to bring it into perfect consonance as a pure major sixth below the starting note, and distributing this minute adjustment as evenly as possible through the fifths thus far tuned. All remaining notes in the series of fifths are tuned as pure major sixths to notes already tuned, as shown in the following diagram. (Note: All octaves of a note are tuned simultaneously on the enharmonium.)

	B#	F##	C##							
begin: →	A	E	B	F#	C#	G#	D#	A#	E#	B#
	G*b*	D*b*	A*b*	E*b*	B*b*	F	C	G	D	(A)
						E*bb*	B*bb*	F*b*	C*b*	(G*b*)

The enharmonium tuning described above is not proposed as a replacement for any existing system; it is simply a practical and very accurate synthesis of two systems having many intervals of nearly identical size despite differing derivations and functions. The accuracy with which the enharmonium is able to represent the values of both Pythagorean and just systems will be apparent from a study of Appendix III, which lists the enharmonium intervals as well as those of Pythagorean and just intonation and equal temperament. (With regard to accuracy, note that the skhisma is about the limit of the ear's ordinary ability to distinguish differences of pitch.[5] Enharmonic pitch differences are between ten and twelve times this amount; the extremely small mistuning of the enharmonium fifth and its pure major third is about 1/9 of this amount.)

II. The Keyboard

A practical enharmonium keyboard should exhibit spatial relationships that correspond in a consistent way to the relationships between tones, so that fingering patterns are uniform in all keys.

As shown below in Diagram A (for clarity, whole-tone rather than fully chromatic scales are shown), a simple two-row keyboard arrangement with enharmonics vertically paired is not possible, for the octave repetition of any given note will always appear on the next lower row. One could, of course, stay on two rows by inverting the enharmonic relationship from octave to octave—i.e., placing sharps above flats in one octave and flats above sharps in the next—but this inconsistency would obviously be confusing.

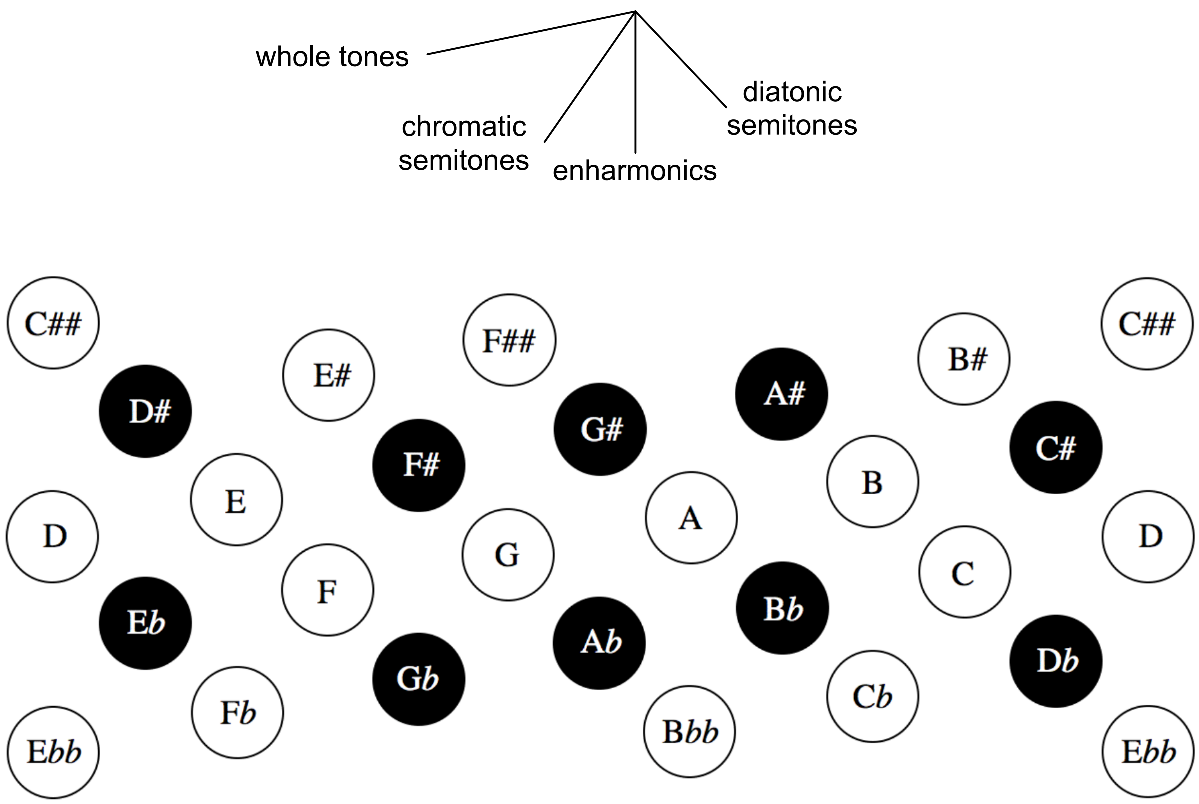
Diagram A: Attempted two-row arrangement of enharmonic whole-tone scales

G#	A#	B#	C##									
A*b*	B*b*	C	D	E	F#	G#	A#	B#	C##			
			E*bb*	F*b*	G*b*	A*b*	B*b*	C	D	E	F#	G#
									E*bb*	F*b*	G*b*	A*b*

[5] See Partch, *Genesis of a Music*, p. 310.

The plan adopted for the enharmonium keyboard is simply to incline the scale upward slightly from left to right, enough so that the octave repetition of a note is always on the same horizontal line. The complete enharmonic chromatic scale, thus arranged, is illustrated and explained in Diagram B.

Diagram B. The enharmonium keyboard plan, with Pythagorean nomenclature for the notes of the enharmonic chromatic scale. To facilitate orientation on the enharmonium keyboard, black and white keys correspond to their equivalents on a conventional keyboard, and the octave span is the same. Note the consistent arrangement of intervals in four directions.



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APPENDIX III

Table of Intervals

This table lists the important intervals of Pythagorean tuning, just intonation, equal temperament, and the enharmonium tuning. Cent values have been calculated from logarithms. Decimal ratios of Pythagorean and just intervals are derived from ratios expressed as fractions; of tempered intervals, from powers of the twelfth root of 2; of enharmonium intervals, from powers of the enharmonium fifth (see Appendix II) reduced by powers of 2 to less than one octave.

Cents	Decimal ratio	Interval Description
0.21708	1.0001254	*Mistuning of enharmonium perfect fifth, 1/9 skhisma (a)*
1.0	1.0005778	Cent, 1/100 of an equal semitone
1.95372	1.0011292	Skhisma (a), 32805/32768
1.95500	1.0011299	Skhisma (b), mistuning of equal-tempered fifth
19.55257	1.0113580	Diaskhisma, 2048/2025
20.85505	1.0121191	*Enharmonium comma*
21.50629	1.0125	Syntonic comma, 81/80
23.46001	1.0136433	Pythagorean comma, 531441/524288
70.45535	1.041536	*Enharmonium small semitone*
70.67243	1.0416667	Just small chromatic semitone, 25/24
90.22500	1.0534979	Pythagorean diatonic semitone, 256/243
91.31040	1.0541586	*Enharmonium semitone*
92.17872	1.0546875	Just large chromatic semitone, 135/128
100.0	1.0594631	EQUAL SEMITONE
111.73129	1.0666667	Just small diatonic semitone, 16/15
112.16545	1.0669342	*Enharmonium semitone*
113.68501	1.0678711	Pythagorean chromatic semitone
133.02049	1.0798646	Enharmonium large semitone
133.23757	1.08	Just large diatonic semitone, 27/25
180.44999	1.1098579	Pythagorean diminished third, 65536/59049
182.40371	1.1111111	Just small whole tone, 10/9
182.62079	1.1112505	*Enharmonium interval*
200.0	1.1224621	EQUAL WHOLE TONE
203.47584	1.1247179	*Enharmonium whole tone*
203.91000	1.125	Whole tone, 9/8
294.13500	1.1851852	Pythagorean minor third, 32/27
294.78624	1.1856312	*Enharmonium interval*
300.0	1.1892071	EQUAL MINOR THIRD
315.64129	1.2	Just *(and enharmonium)* minor third, 6/5
317.59501	1.2013548	Pythagorean augmented second, 19683/16384
384.35999	1.2485903	Pythagorean diminished fourth, 8192/6561
386.09663	1.2498434	*Enharmonium interval*
386.31371	1.25	Just major third, 5/4
400.0	1.2599211	EQUAL MAJOR THIRD
406.95168	1.2649903	*Enharmonium interval*
407.82000	1.265625	Pythagorean major third, 81/64

Cents	Decimal ratio	Interval Description
498.04500	1.3333333	Pure perfect fourth, 4/3
498.26208	1.3335005	*Enharmonium perfect fourth*
500.0	1.3348399	EQUAL PERFECT FOURTH
519.11713	1.3496615	*Enharmonium interval*
519.55129	1.35	Just acute fourth, 27/20
521.50501	1.3515243	Pythagorean augmented third, 177147/131072
588.26999	1.4046639	Pythagorean diminished fifth, 1024/729
589.57248	1.4057212	*Enharmonium interval*
590.22372	1.40625	Just augmented fourth, 45/32
600.0	1.4142136	EQUAL TRITONE (augmented fourth, diminished fifth)
609.77628	1.4222222	Just diminished fifth, 64/45
610.42752	1.4227573	*Enharmonium interval*
611.73001	1.4238281	Pythagorean augmented fourth, 729/512
678.49500	1.4798106	Pythagorean diminished sixth, 262144/177147
680.44871	1.4814815	Just grave fifth, 40/27
680.88287	1.4818530	*Enharmonium interval*
700.0	1.4983071	EQUAL PERFECT FIFTH
701.73792	1.4998119	*Enharmonium perfect fifth*
701.95500	1.5	Pure perfect fifth, 3/2
792.18000	1.5802469	Pythagorean minor sixth, 128/81
793.04832	1.5810397	*Enharmonium interval*
800.0	1.5874011	EQUAL MINOR SIXTH, augmented fifth
813.68629	1.6	Just minor sixth, 8/5
813.90337	1.6002006	*Enharmonium interval*
815.64001	1.6018066	Pythagorean augmented fifth, 6561/4096
882.40499	1.6647869	Pythagorean diminished seventh, 32768/19683
884.35871	1.6666667	Just *(and enharmonium)* major sixth, 5/3
900.0	1.6817928	EQUAL MAJOR SIXTH, diminished seventh
905.21376	1.6868654	*Enharmonium interval*
905.86500	1.6875	Pythagorean major sixth, 27/16
968.82591	1.75	Natural (harmonic) minor seventh, 7/4
972.63000	1.7538495	Pythagorean double-diminished octave, 8388608/4782969
975.66911	1.756931	*Enharmonium interval*
976.53743	1.7578125	Just augmented sixth, 225/128
996.09000	1.7777778	Minor seventh, 16/9
996.52416	1.7782237	*Enharmonium interval*
1000.0	1.7817975	EQUAL MINOR SEVENTH, augmented sixth
1017.37921	1.7997743	*Enharmonium interval*
1017.59729	1.8	Just acute minor seventh, 9/5
1019.55001	1.8020325	Pythagorean augmented sixth, 59049/32768
1087.83455	1.8745299	*Enharmonium interval*
1088.26871	1.875	Just major seventh, 15/8
1100.0	1.8877487	EQUAL MAJOR SEVENTH
1108.68960	1.8972478	*Enharmonium interval*
1109.77500	1.8984375	Pythagorean major seventh, 243/128
1200.0	2.0	OCTAVE, 2/1

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