THE PLEASURES OF MATHEMATICS

F. W. Niedenfuhr, Professor of Engineering Mechanics at Ohio State University, lures the amateur scientist into an encounter with integral calculus

IT MAY COME as something of a surprise to amateurs to learn that they often skate dangerously close to the edge of integral calculus. Integral calculus is not nearly so formidable as it sounds. It is a study, at least in part, of the problem of measuring area. In view of the fun that can be got out of mathematics, and of the understanding of advanced work in science which it can afford, it is unfortunate that more amateurs do not devote some of their time to the subject. Experiments with problems of area can make an interesting starting point. I will give an example of a difficult problem later on, but first let us look into an easier case.

Imagine that we have drawn a simple closed curve on a sheet of writing paper. (Simple means that the curve does not cross itself.) This curve marks out an area on our paper. How many ways can you think of to find this area?

The problem is interesting to me because I like to watch the ways in which students at various stages of sophistication attempt to solve it. A third-year college student may begin by trying to write the equations of the curve. A mature mathematician will ask: "How accurately do you want to know the area?" A graduate student in mathematics will sometimes protest that he is not sure he understands the problem, and that anyway it probably cannot be solved. An engineer may admit that he once knew how to find the area but has now forgotten, or he may produce a machine called a planimeter and proceed to measure the area for you. Usually he will not know why this machine works, but he is pretty sure that it does. But don't press the poor fellow — he has another job to do.

I saw a very clever (and most significant) solution to this problem at a model-airplane meet some years ago. The contest rules required that the models have a fuselage whose cross-sectional area was not less than a certain minimum. This area was easy to check in the good old days when all the models had rectangular cross sections, but with the advent of more streamlined shapes the judges began to have trouble making sure the rules were being followed.

They finally decided to find the required area by first having an accurate drawing of it, and then cutting out the drawing and weighing it. Since the weight per unit area of the paper was known, the area of the cut-out drawing was easy to obtain. Now this solution to the area problem is a splendid example of applied integral calculus. It is a little surprising then, that people who have actually studied calculus will laugh at the method, or dismiss it as impractical. Yet when an accurate balance or scale is at hand it is the quickest way to determine an area. (How many ways can you think of for improvising a balance to "weigh areas"?)

Another obvious way to find an area is to draw the figure on graph paper and count the number of little squares inside the simple closed curve. Then if we know the area of each square we find the total area by multiplying the area per square by the number of squares enclosed by the curve. Of course near the edges of the figure we will have to count partial squares, and some error will be introduced each time we estimate the size of such a square.

But the total error will generally be small for two reasons. First we will sometimes overestimate and sometimes underestimate the area of a partial square, and our errors from this source will tend to cancel out. Second, the human eye is an excellent judge of the relative sizes of small areas. This process is not so tedious as might be imagined, because on the interior of the figure we can count great blocks of squares at once, rather than each square individually. I find that making drawings on 8 ½ X 11-inch paper with quarter-inch squares printed on it provides excellent accuracy and is not too time-consuming.

A variation on the system of counting squares is the Monte Carlo method. You might like to try this one experimentally. Draw the area on a piece of paper again, and put your finger down at random. One of four things will happen: (1) your finger will come down on the paper inside the unknown area, (2) it will come down outside the area, (3) it will come down on the boundary of the area, or (4) it will miss the paper entirely. Now on a separate tally sheet keep track of the results as follows: In the first case (your finger lands inside the area) write "Yes" on the tally sheet. In the second case write "No" on the sheet. In the third and fourth cases do not write on the sheet at all. After a large number of tallies have been made you can find the unknown area by multiplying the total area of the paper by the number of "'Yes" tallies divided by the number of "Yes" plus "No" tallies. The accuracy of the answer will depend on two things. First, the number of tallies must be large; second, you must put your finger down in a random manner each time. Obviously if you always put your finger down outside the given area, you would have no "Yes" tallies, and the formula given above would indicate that the unknown area is zero.

Pursued by hand, the Monte Carlo method will only lead to bruised thumbs and poor estimates of the area, but it does appear to be a useful method when automatic machines can be devised to make and record a large number of tallies. Machines have been constructed which integrate (calculate areas) by this method, but they are handicapped by the difficulty of providing random numbers which tell the machine how to "put its finger down."

Any mechanical device to produce random numbers will be subject to wear, and this wear introduces a bias in favor of a particular number. For a time it was thought the sequence of digits in *pi* (3.14159...) would be random, but this is not the case. There still is no completely satisfactory way to produce random numbers. In spite of this practical difficulty, the Monte Carlo method of integration holds great promise.

There is still another way in which the original problem can be solved. Suppose our unknown area has been divided up into a large number of narrow strips by equally spaced vertical lines drawn on the paper [*Fig. 1*]. By itself each strip differs little from a long, narrow rectangle. Suppose we have numbered each strip for identification purposes and measured its length. If the lengths of the strips are L_1, L_2, L_3 and so on, the following is an obvious formula for the area:

$$\text{Area} = (L_1 + L_2 + L_3 \ldots) \times \boldsymbol{B}$$

B is the width of the strips. Now the ends of each strip will not be rectangular but tapered or cut on the bias. If a strip is rather wide, you may have difficulty in deciding its length, and the accuracy of the final result will depend on how well you define the length of each strip. If the strips are narrow, however, it will be easy to decide the length. Imagine, for instance, that each strip is as narrow as the thickness of the paper. If the strips were literally cut apart, we could measure the length of a number of threads. This would be tedious but not difficult. Thus the accuracy of the formula increases as *B* becomes very small, and as the

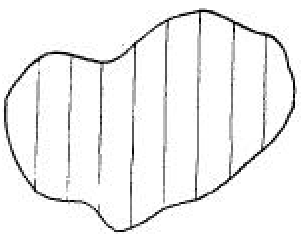


Fig. 1
Division of irregular area into strips of uniform width

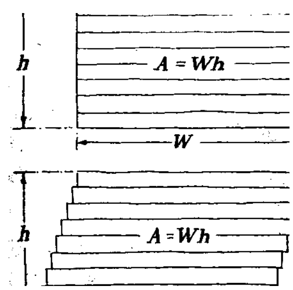


Fig. 2
Laminated area *(top)* does not change in size when canted *(bottom)*

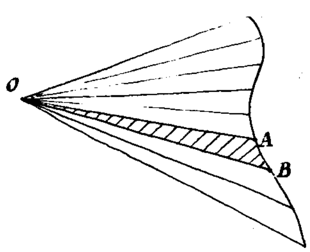


Fig. 3
Irregular area divided by rays of equal angle

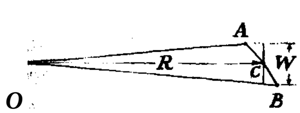
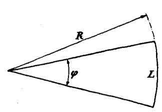


Fig. 4
Element of irregular area divided by rays of equal angle

number of strips increases. The "fundamental theorem" of integral calculus is based on this formula.

The area given by formula will not be changed if the strips are moved with respect to one another. This explains why the areas of the parallelogram and the rectangle in the accompanying illustration [*Fig. 2*] are the same. The parallelogram is just the rectangle with its strips pushed over a little. The pushing process does not change either the length or width of a strip.

Another process for finding areas is to divide the unknown area into a large number of triangles, find the area of each triangle, and add up the areas. We split the area by drawing rays, each of which has the same angular relationship to the others [*Fig. 3*]. Now look at a typical element of this area [*Fig. 4*]. Let C be the center of AB, and *R* be the distance from 0 to C. Imagine that we swing an arc of radius *R (OC)* between the lines *OA* and *OB*. If the angle *AOB* is very small, this arc will appear to be practically a straight line of length W. The area of the triangle is then one-half $R \times W$.



<u>Fig. 5</u>
Angle measured by graphic method

If the angle *AOB* is denoted by the symbol (dj), you see that the two lengths R and W determine the angle. In fact, we may say by definition that $(dj) = W/R$. You are familiar with measuring angles in degrees, but this new method of measuring angles is generally more useful in mathematics. The units of this measurement are called "radians." For instance, if in the above example $R = 3$ inches and W = .3 inch, then (dj) *is* an angle of, .1 radians, which is equivalent to just under six degrees.

Measurement of larger angles in radians may be done by drawing a circular sector [*Fig. 5*]. Let R be the radius of the sector, and *L* be the length of the arc. The angle j, expressed in radians, is $j = L/R$. If we keep increasing the angle j, the sector opens up into a circle, and the angle j becomes 360 degrees.

How many radians is this? The "arc length" L has become equal to the circumference of the circle which is 2πR. Then j = L/R = 2πR/R = 2π. Thus 2π radians equals 360 degrees. The advantage of using the radian measure for angles lies in the fact that if we say "angle equals arc length over radius," we may also say "arc length equals angle times radius."

Back to our little triangle. Since (dφ) = W/R, or W= B (dφ), the area of the little triangle is ½RW = ½ R^2 (dφ). Let R_1, R_2, R_3 . . . be the radii of the small triangular "slices" of the big area. Then, since each "slice" has the same central angle (dφ), the big area is given by the formula:

$$A = \tfrac{1}{2} (R_1^2 + R_2^2 + R_3^2 + \ldots) (dj)$$

This represents the sum of the areas of the little triangles. This formula too is very close to a calculus formula. Its accuracy will increase as (*dj*) decreases and the number of terms increases.

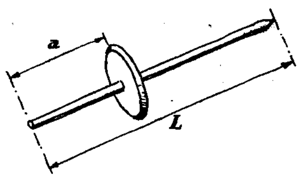


Fig. 6
Basic elements of planimeter

You now have enough information to build a machine to measure areas (a planimeter). Imagine a straight bar which serves as the axle of a knife-edged wheel [Fig. 6] Now imagine that this bar moves a small amount parallel to the plane of the paper, while the wheel rolls and slides on the paper. Let the bar move from BC to B'C' [Fig. 7}. This motion could be accomplished by first moving the bar parallel to itself from BC to B'C" and

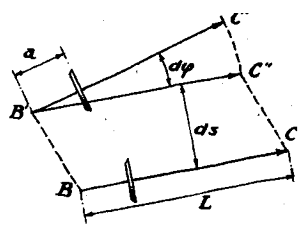


Fig. 7
Geometry of planimeter action

then rotating it about *B'* until it reaches *B'C'*. How far would the wheel roll during this motion? In moving from BC to *B'C"*, the wheel rolls (and slides sideways a little) a distance (*ds*), and in moving from *B'C"* to *B'C'* the wheel rolls a distance *a* (*dj*). [Here again (*dj*) is measured in radians, and is a small angle.] So the total distance the wheel rolls is (*dp*), where (*dp*) = *(ds) + a* (*dj*).

What area did the bar sweep out? In moving from BC *to B'C"* the area covered was *L*(*ds*), and in moving from *B'C" to B'C'* the area covered was ½ L^2 (*dj*), so if the total area swept out is called (dA):

$$(dA) = L(ds) + \tfrac{1}{2} L^2 (dj)$$

Combining the last two equations:

$$\begin{aligned}(dA) &= L\,(dp) - La\,(dj) + \tfrac{1}{2} L^2 (dj)\\ &= L\,(dp) + (\tfrac{1}{2}L^2 - La)\,(dj)\end{aligned}$$

Now you should know, if you have not already guessed, that in the notation of calculus if *x* is any quantity, the symbol (*dx*) stands for a little bit of *x*. For instance, if A is an area, (*dA*) is a very small "slice" of that area. If *p* is a distance, (*dp*) is a short step along the way.

Having mastered this much calculus, we may now play a trick on our little wheel and axle—or tracer arm, as it is properly called. We attach another bar to it at point *B.* This second bar is called a polar arm. The polar arm has one end hinged to the tracer arm at *B,* and one end fixed (but free to pivot) at point 0 [Fig. 8].

Now trace around the circumference of an area with the tracer point *C* . The area swept out by *BC* will be the area we are attempting to measure plus the area hatched in the illustration. But the hatched area will be covered twice, once with the wheel rolling forward and once with the wheel rolling backward, so it cancels out. If we consider the total area swept out by the tracer arm

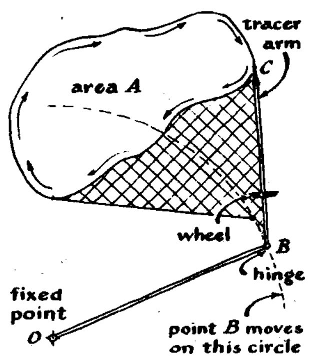


Fig. 8
Demonstration of integration by use of the planimeter

as being the sum of a large number of very small areas (dA), as described above, we may write:

$$
\begin{aligned}
A &= (dA)_1 + (dA)_2 + (dA)_3 + \ldots \\
&= L\,[(dp)_1 + [(dp)_2 + \ldots] + \\
&\quad (\tfrac{1}{2}L^2 - La)\,[(dj)_1 + (dj)_2 + \ldots]
\end{aligned}
$$

Now we must interpret each of these sums. The total distance (*p)* through which the wheel rolled is:

$$p = [(dp)_1 + (dp)_2 + \ldots]$$

The total angle through which the tracer arm BC turned is:

$$j = [(d\varphi)_1 + [(d\varphi)_2 + \ldots]$$

But since the polar arm forced the tracer arm to return to its exact starting point, the total angle turned is zero. Thus $A = Lp$ (*p*, again, is the distance the wheel rolled). This is easily obtained from scale markings on the rim of the wheel.

The little instrument we have been discussing is called a polar planimeter [Fig. 9]. There are several models on the market —all rather expensive for beginners, I fear. You can have more fun constructing your own. An accurate polar planimeter is a precision machine. If you are not quite up to fine mechanical

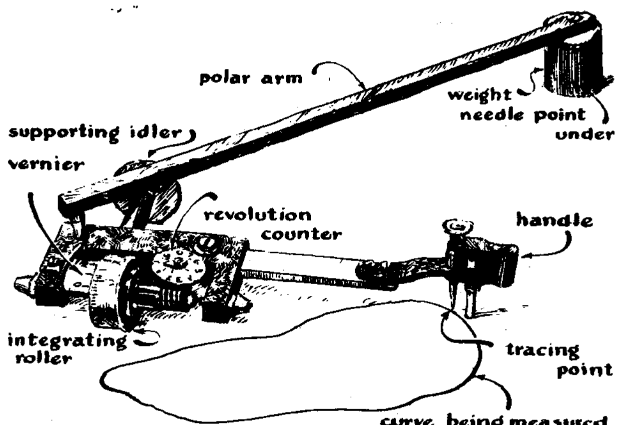


Fig.9
Practical form of the planimeter

work, there is another way out. Your pocketknife can be used as a satisfactory, if approximate, planimeter. Open both blades as shown in the upper portion of the accompanying drawing [*Fig. 10*]. Make sure that blade *B* makes contact on the cutting edge, and that blade C makes contact at its point when the opened knife is held upright on a table. Determine the distance L.

Now pick an area to be measured. The longest diameter of the area should be considerably shorter than L; say, no more than half as long. Locate the center of the area approximately. Draw a straight line (to be used as a reference line) outward from the center of area *[see lower diagram. Fig. 10].* Now hold the knife so that point C is at the center of the area, and *B* is on the reference line. Holding blade C, push the knife along the reference line until the point of C is on the boundary of the curve. Then, keeping the knife upright, guide point C completely around the area, tracing out the boundary line, and finally pull it back to the center. Meantime the supporting blade B will have been riding freely on the paper. When you return to the starting position,

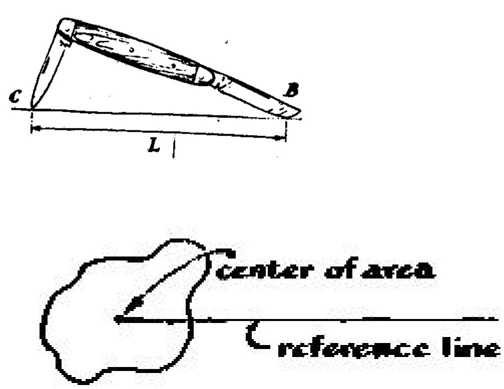


Fig. 10
The jackknife in position for use as a planimeter *(top)*. Irregular area to be measured together with reference line *(bottom).*

blade *B* will no longer be on the reference line, and the line of the knife, *BC*, will make an angle with the reference line. Call this angle j (measured in radians). The area you have traced around is then given approximately by the formula $A = L^2 j$. This is so easy to do that it is worthwhile trying it on a few known areas just to see how good your pocketknife really is.

If it is inconvenient to measure the angle directly in radians, measure it in degrees, multiply by *2p* and divide by 360, in accordance with the definition of a radian given above. This sliding type of planimeter is often called a "hatchet planimeter."

Now for the more difficult problem I promised earlier. It would seem natural, in view of all the foregoing, to seek the area of a curved surface by enclosing the surface in a polyhedron of many sides and then adding up the areas of the faces of the polyhedron. For example, the area of a cylinder can be obtained approximately by adding up the areas of the faces of an octagonal box in contact with the cylinder [*Fig. 11*]. The result would be more accurate if instead of an octagonal box we used a box with many more sides. Just to show that this is not always such an easy process, consider the following. Cut the cylinder along a vertical line and unroll it into a flat sheet. The area of this sheet

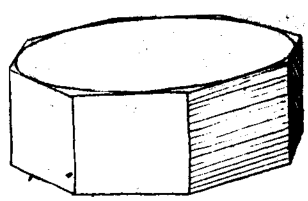


Fig. 11
Area of cylinder approximated by octagonal box

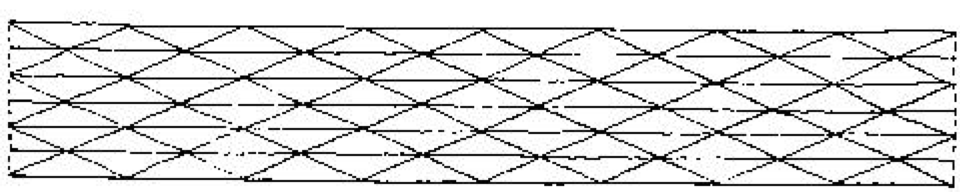


Fig. 12
Cylinder analyzed into rectangles and triangles

is the same as the area of the cylinder. Divide the flat sheet into rectangles and triangles as shown [*Fig. 12*].

Now roll up this sheet to form a cylinder again. Let each vertex of each of the triangles stay on the surface of the cylinder. By connecting these vertices with straight lines we form a polyhedron with a large number of triangular faces which is inscribed in the cylinder. The sides of the triangles are not on the surface of the cylinder, but run inside it from one surface point to another. It is tempting to assume that the area of the polyhedron more and more closely approximates the area of the cylinder as the number of triangles is increased, that is, as the grid of rectangles and triangles becomes finer and finer.

This is in fact not true. If the grid is chosen properly, the polyhedron can be made to approach a kind of Japanese-lantern shape [*Fig. 12*]. The area of such a polyhedron can be made very much *larger* than the area of the circumscribing cylinder. The reason for this is that the planes of the triangles slant in and out with respect to the surface of the cylinder. If the number of vertical divisions is very large as compared to the number of horizontal divisions, the triangles become almost perpendicular to the curved surface. Thus the area of these triangles bears no particular relation to the area of the cylinder.

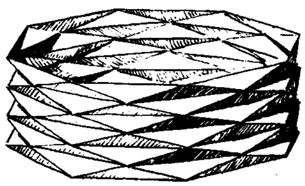


Fig. 12
Cylinder of indefinitely large surface area unrelated to its volume

(From “The SCIENTIFIC AMERICAN Book of Projects for The Amateur Scientist”, by C.L. Stong, Simon and Schuster, New York 1960)